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TRACE AND TORSION IN THE THEORY OF FLOWS

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§ 0. INTRODUCTION

THIS paper is about new uses of algebra in dynamical systems. The algebra involved is elementary algebraic K-theory and Hochschild homology. Our main algebraic tools are recalled in § 1(A).

Our goal is to show that certain invariants of a flow can be computed algebraically as traces, and that some of these traces come naturally from interesting new K_1 and torsion invariants of the flow. Indeed, torsion invariants of "Reidemeister type" have been known in dynamics for some time [7]; our theory recovers these; see § 5(B) and Corollary 7.10. Whereas the invariants in [7] are "commutative" in the sense of being associated with the universal *abelian* cover of the manifold, our invariants are "non-commutative", being associated with the universal cover.

This paper is an application to dynamics of our paper [9] where we developed one-parameter Lefschetz–Nielsen fixed point theory for a homotopy $F: X \times [a, b] \to X$ where X is a finite CW complex. That theory is summarized in § 1(C). The *fixed point set* of F is the set $\mathrm{Fix}(F) = \{(x, t) \in X \times [a, b] \mid F(x, t) = x\}$. Pick a basepoint $v \in X$, and write $G \equiv \pi_1(X, v)$. In [9], we introduced "traces", which we denoted by $L(F)$ and $R(F)$. These "traces" are computable invariants, depending only on the homotopy class of F rel $X \times \{a, b\}$. The invariant $L(F)$, analogous to the Lefschetz number, takes values in the homology $H_1(G) \equiv G_{\mathrm{ab}}$ and the invariant $R(F)$, analogous to the "Reidemeister trace" of classical Nielsen fixed point theory, takes values in the Hochschild homology group $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$ where $\phi: \mathbb{Z}G \to \mathbb{Z}G$ is a ring endomorphism induced by F and $(\mathbb{Z}G)^{\phi}$ is $\mathbb{Z}G$ as an abelian group and endowed with a suitable $\mathbb{Z}G - \mathbb{Z}G$ bimodule structure depending on ϕ (defined in § 1(A) below).

Given a smooth flow $\Phi: M \times \mathbb{R} \to M$ on a closed oriented manifold one may regard any finite portion of Φ as a homotopy. Writing $F = \Phi| : M \times [a, b] \to M$, one would expect our traces $L(F)$ and $R(F)$ to have recognizable dynamical meaning, and indeed they do. When $a > 0$, $L(F)$ detects the *Fuller homology class*, derived from Fuller's index theory [8], of the closed orbits occurring in that portion of the flow. The invariant $R(F)$ does the same thing in a more precise way taking into full account the important role of the fundamental group. See Theorems 2.5 and 2.7. These theorems should be viewed as "Lefschetz trace formulae" for flows; roughly speaking, they assert that the algebraic traces "see" the closed orbits homologically.

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A different invariant of a non-singular flow Φ is detected by our traces when the "time" interval $[a, b]$ is $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$ is such that Φ has no fixed points in $[-\varepsilon, \varepsilon]$ other than those at 0. Recall the Poincaré–Hopf index theorem which asserts that if $\mathscr{X}$ is a smooth vector field on the closed oriented manifold M with only finitely many zeros then the global sum of the indices of $\mathscr{X}$ equals the Euler characteristic of M. The global sum of the indices of $\mathscr{X}$ can be viewed as a geometric definition of the Poincaré dual of the Euler class of the tangent bundle of M; on the other hand the Euler characteristic can be defined as an alternating sum of traces on homology (or cellular chains). Now, suppose that $\mathscr{X}$ is a non-singular vector field on M and that Φ is the flow on M determined by $\mathscr{X}$. The main theorem of § 3 is:

THEOREM (One-parameter Poincaré–Hopf Theorem). *The homology class* $-L(\Phi|_{M\times[-\varepsilon,\varepsilon]}) \in H_1(M) \equiv H_1(G)$ *is the Poincaré dual of the Euler class of the normal bundle of the flow* Φ.

In § 4 we apply our theory to a suspension semiflow, i.e. the natural semiflow Φ: $Y \times [0, \infty) \to Y$ on the mapping torus, $Y = T(X, f)$, of a continuous map f: $X \to X$ which is a π_1-*equivalence* (i.e. f induces an isomorphism of fundamental groups). Of course, one expects a connection with the classical Nielsen fixed point theory of f; geometric motivation for this connection is provided in § 4(B) and § 4(C). The main computational result of § 4 is Theorem 4.2 which is an explicit formula, in case f is cellular, for the one-parameter trace $R(\Phi|_{Y\times[0,\ell+1]})$ where l is a positive integer. Up to this point we have dealt with only a finite portion of the semiflow; however, the form of the answer obtained in Theorem 4.2 becomes particularly striking if we allow $\ell \to \infty$ in the appropriate algebraic context. It suggests that there ought to be an algebraic trace which "sees" *all* the closed orbits in the *non-compact* time interval $(0, \infty)$. This observation motivates much of § 5 and § 6.

Consider a continuous self-map f: $X \to X$ of a finite complex. The "Lefschetz zeta function" of f is the formal power series $\zeta_f(t) = \exp(\sum_{n=1}^{\infty} \frac{1}{n} L(f^n) t^n) \in \mathbb{Q}[[t]]$ where $L(f^n)$ is the Lefschetz number of the n-th iterate of f. Originally introduced by Weil in a number theoretic setting, it is an efficient means of keeping track of the homological data associated to the fixed points of the iterates of f. Since we wish to take account of the full influence of the (typically non-commutative and infinite) fundamental group of X, we seek a "non-commutative" substitute for $\zeta_f(t)$. This requires some new algebra which we now summarize; details are given in § 5.

Let S be an associative ring with unit and θ: $S \to S$ a ring homomorphism. We do not assume S is commutative; in typical applications S will be the integral group ring of a group H and θ will be induced by an automorphism of H. Let (C, ∂) be a finitely generated chain complex of right S-modules such that each C_i is free with a given basis. Suppose f_*: $C \to C$ is a θ-*homomorphism*, i.e. a degree 0 homomorphism of the underlying graded abelian groups such that $f_i(mr) = f_i(m)\theta(r)$ for $m \in C_i$ and $r \in S$. The *Reidemeister trace* of f_*, which we denote by $R(f_*)$, is the element of $HH_0(S, S^\theta)$, the 0-th Hochschild homology of S with coefficients in the bimodule S^θ (see § 1(A)), represented by the alternating sum of traces: $\sum_j (-1)^j \operatorname{trace}[f_j]$ where $[f_j]$ is the matrix of f_j with respect to the given basis. For $n \geq 1$ the n-th iterate of f_*, denoted f_*^n, is a θ^n-homomorphism. Note that $R(f_*^n) \in HH_0(S, S^{\theta^n})$. Let $\bar{R}(f_*^n)$ denote the image of $R(f_*^n)$ in $HH_0(S, S^{\theta^n})_\theta$, the quotient group of co-invariants of a natural action induced by θ on $HH_0(S, S^{\theta^n})$. Geometric motivation for the passage from $R(f_*^n)$ to $\bar{R}(f_*^n)$ is given in § 4(B).

Definition. (See 5.1.) The *Lefschetz-Nielsen series* of a θ-homomorphism f_*: $C \to C$ is given by $\mathrm{LN}(f_*) = (\bar{R}(f_*^n))_{n=1}^{\infty} \in \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$.

Define the *θ-twisted power series ring*, denoted by $S_\theta[[t]]$, as follows: elements of $S_\theta[[t]]$ are formal series $\sum_{i=0}^{\infty} u_i t^i$ where $u_i \in S$ and t is an indeterminate, and multiplication is defined by $(\sum_{i=0}^{\infty} u_i t^i)(\sum_{j=0}^{\infty} v_j t^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} u_i \theta^i(v_j))t^k$.

Let I_{n_i} be an identity matrix of the same size as $[f_i]$. Since the matrix $I_{n_i} - [f_i]t$ is invertible over $S_\theta[[t]]$, we may regard $I_{n_i} - [f_i]t$ as an element of the infinite general linear group over $S_\theta[[t]]$ and so we can define $\Delta(f_*) \in K_1(S_\theta[[t]])$ as follows:

Definition. (See 5.2.) $\Delta(f_*) \in K_1(S_\theta[[t]])$ is the element represented by the finite alternating product $\prod_{i \geq 0}(I_{n_i} - [f_i]t)^{(-1)^{i+1}}$.

The main results of § 5(A) are Theorems 5.6 and 5.10. Theorem 5.6 asserts that $\mathrm{LN}(f_*) = P_+ \mathrm{DT}(\Delta(f_*))$ where $\mathrm{DT}: K_1(S_\theta[[t]]) \to HH_1(S_\theta[[t]])$ is the *Dennis trace* homomorphism (see § 1(A)) and $P_+ : HH_1(S_\theta[[t]]) \to \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$ is a naturally defined homomorphism. Thus, $\Delta(f_*)$ determines the Lefschetz–Nielsen series of f_*. There is a "completed" version of the Hochschild homology of a twisted power series ring (see Definition 5.5), denoted by $\widehat{HH}_*(S_\theta[[t]])$, and a natural homomorphism $HH_*(S_\theta[[t]]) \to \widehat{HH}_*(S_\theta[[t]])$. The homomorphism P_+ factors as $HH_1(S_\theta[[t]]) \to \widehat{HH}_1(S_\theta[[t]]) \xrightarrow{\hat{P}_+} \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$. Suppose that the ring S is also a vector space over $\mathbb{Q}$ so that division by a nonzero integer is possible. Then Theorem 5.10 asserts the sharpened identity: $\mathrm{Lg}(\mathrm{LN}(f_*)) = \widehat{\mathrm{DT}}(\Delta(f_*))$, where Lg, defined by a formula reminiscent of a logarithm, is a right inverse for $\hat{P}_+$, i.e. $\hat{P}_+ \mathrm{Lg}$ is the identity, and $\widehat{\mathrm{DT}}$ is the composite $K_1(S_\theta[[t]]) \xrightarrow{\mathrm{DT}} HH_1(S_\theta[[t]]) \to \widehat{HH}_1(S_\theta[[t]])$. Theorems 5.6 and 5.10 deserve to be called "rationality theorems" (akin to "rationality of the Lefschetz zeta function") as they show that the Lefschetz–Nielsen series can be computed from a finite alternating product of "characteristic polynomials" defining an element of $K_1(S_\theta[[t]])$. The Hochschild homology element $\mathrm{DT}(\Delta(f_*)) \in HH_1(S_\theta[[t]])$ should be thought of as a "non-commutative" substitute for the "Lefschetz zeta function". Reinforcing this point of view, we show in § 5(B) how the well known rationality of certain algebraic zeta functions used in fixed point theory and dynamics can be derived from Theorem 5.10. The main results in § 5(B) are Theorem 5.13 and Examples 5.15 and 5.16 in which we recover as corollaries results from [13] and [7].

In § 6, the algebra of § 5 is applied to define a new K_1-type invariant, the *total Lefschetz–Nielsen invariant*, for any continuous map $f: X \to X$ of a finite connected CW complex X. When f is cellular, this invariant, denoted by $\Delta(f)$, is defined to be $\Delta(\tilde{f}_*) \in K_1(\mathbb{Z}H_\theta[[t]])$ where H is the fundamental group of X, $\theta: H \to H$ is the endomorphism determined by f (and a choice of basepoint and basepath) and $\tilde{f}_*: C_*(\tilde{X}) \to C_*(\tilde{X})$ is the θ-homomorphism induced by f on the cellular chains of the universal cover, $\tilde{X}$, of X (see Definition 6.1). $\Delta(f)$ has nice invariance properties (see Propositions 6.3, 6.4, and 6.6). According to Corollary 6.7, $\Delta(f)$ determines the classical Nielsen fixed point theory of the iterates of f, up to the ambiguity of passing to the co-invariants of the θ-action on Hochschild homology. The main computational result of § 6 is Theorem 6.9, which gives a formula relating the Dennis trace of $\Delta(f)$ to the one-parameter trace $R(\Gamma)$ where f is a cellular π_1-equivalence and Γ is the restriction of the natural semiflow on the mapping torus $Y = T(X, f)$ to $Y \times [0, \ell + 1]$, ℓ being a positive integer. Using this result, we are able to elucidate the algebraic relationships between the various intersection invariants discussed in § 4(C); see Theorem 6.12.

In § 7 we introduce a new kind of *torsion invariant* (in the sense of Whitehead torsion) for certain finite connected CW complexes whose fundamental groups are semidirect products with an infinite cyclic group.

Let Y be a finite connected CW complex with fundamental group G. Suppose we are given an epimorphism $\psi: G \to T$ where T is infinite cyclic. In order to formulate our results, we make use of the *Novikov ring* of (G, ψ), denoted by $\widehat{\mathbb{Z}G}^+$, which is obtained from $\mathbb{Z}G$ by completion with respect to a natural valuation determined by ψ. See Definition 7.1 for an explicit description of $\widehat{\mathbb{Z}G}^+$. This ring also has been used in [16], [17] and elsewhere.

Recall that for any left $\mathbb{Z}G$-module M the homology $H_*(Y, M)$ is, by definition, $H_*(C_*(\tilde{Y})\otimes_{\mathbb{Z}G} M)$ where $C_*(\tilde{Y})$ is the cellular chain complex of the universal cover of Y. Suppose, $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$. Then the *torsion* of (Y, ψ), denoted by $\tau(Y, \psi)$, is the element of $K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ given by the torsion of the acyclic chain complex $C(\tilde{Y})\otimes_{\mathbb{Z}G}\widehat{\mathbb{Z}G}^+$ where $K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ is the cokernel of the natural map $\pm G \to K_1(\widehat{\mathbb{Z}G}^+)$ (see Definition 7.2). There is a "difference formula" relating the torsion $\tau(h) \in Wh_1(G)$ of a homotopy equivalence $h: Z \to Y$ to the torsions $\tau(Y, \psi)$ and $\tau(Z, \psi')$ when the latter are defined. Let j_*: $Wh_1(G) \to K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ be the homomorphism induced by the inclusion of rings $j: \mathbb{Z}G \subsetneq \widehat{\mathbb{Z}G}^+$.

THEOREM. (*See* 7.3.) $j_*(\tau(h)) = \tau(Y, \psi) - h_*(\tau(Z, \psi'))$. *In particular, if h is a simple homotopy equivalence then* $\tau(Y, \psi) = h_*(\tau(Z, \psi'))$.

Let X be a finite connected complex with fundamental group H and let f: $X \to X$ be a π_1-equivalence inducing an automorphism θ: $H \to H$. Proposition 7.5 asserts that the mapping torus, $Y = T(X, f)$ of f satisfies $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$; in particular, $\tau(Y, \psi)$ is defined. Here ψ: $G \to T$ is induced from the natural map $Y \to S^1$. In that case, the total Lefschetz–Nielsen invariant of f and the torsion invariant of (Y, ψ), are related as follows:

THEOREM. (*See* 7.6.) $i_*(\Delta(f)) = -\tau(Y, \psi)$ *where* i_* *is induced by the inclusion of rings* $i: \mathbb{Z}H_\theta[[t]] \subsetneq \widehat{\mathbb{Z}G}^+$.

The torsion invariant $\tau(Y, \psi)$ can be thought of as a "non-commutative" generalization of the "Alexander invariant" of [13] or of the "Alexander quotient" of [7] as follows. Let P be a finitely generated abelian group and ρ: $G \to P$ a surjective homomorphism. Assume that ψ factors through ρ, i.e. there exists ψ': $P \to T$ such that $\psi = \psi'\rho$. Let $\bar{Y}$ be the covering space of Y, corresponding to $\ker(\rho) \subset G$. Let $C_*(\bar{Y})$ be the cellular chain complex of $\bar{Y}$. We regard $C_*(\bar{Y})$ as a complex of right modules over the ring $\mathbb{Z}P$. Let $\mathbb{Z}P_N$ be the localization of $\mathbb{Z}P$ at its multiplicative group of non-zero divisors.

Suppose $C' \equiv C_*(\bar{Y})\otimes_{\mathbb{Z}P}\mathbb{Z}P_N$ is acyclic. In [7], Fried defined the *Alexander quotient*, denoted by $\mathrm{ALEX}_P(Y)$, to be the element $-\tau(C') \in K_1^{\pm P}(\mathbb{Z}P_N)$ where $\tau(C')$ is the torsion of C'. (Fried prefers to regard $\mathrm{ALEX}_P(Y)$ as an element of (units in $\mathbb{Z}P_N$)/($\pm P$) by applying the determinant, but the above formulation is better adapted to our purposes). The inclusion of rings i: $\mathbb{Z}P_N \subsetneq \widehat{\mathbb{Z}P}_N^+$ induces a homomorphism i_*: $K_1^{\pm P}(\mathbb{Z}P_N) \to K_1^{\pm P}(\widehat{\mathbb{Z}P}_N^+)$; ρ induces a homomorphism $\bar{\rho}$: $K_1^{\pm G}(S_\theta[[t]]) \to K_1^{\pm P}(\widehat{\mathbb{Z}P}_N^+)$.

THEOREM. (*See* 7.9.) *Suppose* $\tau(Y, \psi)$ *and* $\mathrm{ALEX}_P(Y)$ *are both defined. Then* $\bar{\rho}_*(\tau(Y, \psi)) = -i_*(\mathrm{ALEX}_P(Y))$.

Since i_* is injective $\tau(Y, \psi)$ determines $\mathrm{ALEX}_P(Y)$ (see Corollary 7.10). We note that when Y is the mapping torus of a π_1-equivalence of a finite complex, $\tau(Y, \psi)$ and $\mathrm{ALEX}_P(Y)$ are both defined.

Our torsion invariant can be applied to obtain a new invariant, which we call the *non-commutative Alexander invariant*, for a fibered knot K in a homology 3-sphere; see Definition 7.11. This invariant can be thought of as a "non-commutative" generalization of the (ideal in $\mathbb{Q}[t, t^{-1}]$ generated by the) Alexander polynomial of K.

§ 1. REVIEW OF HOCHSCHILD HOMOLOGY AND ONE-PARAMETER FIXED POINT THEORY

For the convenience of the reader, we give a summary of the parametrized fixed point theory of [9] and some background material concerning Hochschild homology and traces.

(*A*) *Algebra*

Let R be a ring and M an R–R bimodule (i.e. a left and right R-module satisfying $(r_1 m)r_2 = r_1(mr_2)$ for all $m \in M$, and $r_1, r_2 \in R$), the *Hochschild chain complex* $\{C_*(R, M), d\}$ consists of $C_n(R, M) = R^{\otimes n} \otimes M$ where $R^{\otimes n}$ is the tensor product of n copies of R and the boundary operator is given by:

$$\begin{aligned} d(r_1 \otimes \cdots \otimes r_n \otimes m) = {} & r_2 \otimes \cdots \otimes r_n \otimes m r_1 \\ & + \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes m \\ & + (-1)^n r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n m. \end{aligned}$$

The tensor products are taken over the integers. The n-th homology of this complex is the n-th *Hochschild homology of* R *with coefficient bimodule* M. It is denoted by $HH_n(R, M)$. If $M = R$ with the standard R–R bimodule structure then we write $HH_n(R)$ for $HH_n(R, M)$. A useful reference for this material is [11].

We will be concerned mainly with HH_1 and HH_0 which are computed from

$$\cdots \to R \otimes R \otimes M \xrightarrow{d} R \otimes M \xrightarrow{d} M$$

$$r_1 \otimes r_2 \otimes m \mapsto r_2 \otimes m r_1 - r_1 r_2 \otimes m + r_1 \otimes r_2 m$$

$$r \otimes m \mapsto mr - rm$$

If R is a ring and ϕ: $R \to R$ is a ring endomorphism, we define R^ϕ to be the R–R bimodule whose underlying abelian group is R and whose bimodule structure is given by: $r \cdot m = rm$ and $m \cdot r = m\phi(r)$ for $r \in R$ and $m \in R^\phi$. As applied below, $R = \mathbb{Z}G$, the integral group ring of a group G and ϕ: $\mathbb{Z}G \to \mathbb{Z}G$ is induced from a group homomorphism ϕ: $G \to G$.

Elements g_1 and g_2 of a group G are *semiconjugate* if and only if there exists $g \in G$ such that $g_1 = g g_2 \phi(g^{-1})$. We write $C(g)$ for the semiconjugacy class containing g and G_ϕ for the set of semiconjugacy classes. The partition of G into the union of its semiconjugacy classes induces a direct sum decomposition of $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ where we think of $g = g_1 \cdots g_n m \in G$ as "marking" a semiconjugacy class. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in $C(g)$ when put into canonical form. For $C \in G_\phi$ let $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ be the subgroup of $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ generated by those generating chains whose markers lie in C. The decomposition $(\mathbb{Z}G)^\phi \cong \bigoplus_{C \in G_\phi} \mathbb{Z}C$ as a direct sum of abelian groups determines a decomposition of chain complexes $C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$. There results a natural isomorphism $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ where the summand $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of C. We call this summand the *C-component*. Given any $\mathbb{Z}G$–$\mathbb{Z}G$ bimodule N let $\bar{N}$ be the left $\mathbb{Z}G$ module whose underlying abelian group is N and whose left module structure is given by $gm = g \cdot m \cdot g^{-1}$. There is a natural isomorphism $HH_*(\mathbb{Z}G, N) \cong H_*(G, \bar{N})$ which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology, see [11, Theorem 1.d]. The decomposition $(\mathbb{Z}G)^\phi \cong \bigoplus_{C \in G_\phi} \mathbb{Z}C$ is a direct sum of left $\mathbb{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{(\mathbb{Z}G)^\phi}) \cong \bigoplus_{C \in G_\phi} H_*(G, \mathbb{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbb{Z}G$ modules $\mathbb{Z}C \cong \mathbb{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = gh\phi(g^{-1})\}$ denotes the *semicentralizer* of $h \in G$. Since $H_*(G, \mathbb{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain a natural isomorphism $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} H_*(Z(g_C))$; furthermore, $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. In particular $HH_0(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \mathbb{Z}G_\phi$, the free abelian group generated by the semiconjugacy classes, and $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} H_1(Z(g_C))$, the direct sum of the abelianizations of the semicentralizers.

Next, we review traces in Hochschild homology. Let R be a ring and M a R–R bimodule. If B is a square matrix over M, its *trace*, denoted by trace (B) is the element $\sum_i B_{ii} \in M$. This element can be viewed as a Hochschild 0-cycle and thus defines an element of $HH_0(R, M)$ which we will denote by $T_0(B)$.

If A is a $n \times m$ matrix over R and B is a $m \times n$ matrix over M, define $A \otimes B$ to be the $n \times n$ matrix with entries in the abelian group $R \otimes M$ given by $(A \otimes B)_{ij} = \sum_k A_{ik} \otimes B_{kj}$. The *trace* of $A \otimes B$ is then $\sum_{jk} A_{jk} \otimes B_{kj} \in R \otimes M$. We interpret this trace as a Hochschild 1-chain. Clearly, it is a cycle if and only if trace(AB) = trace(BA), in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(R, M)$.

We recall the definition of K_1 of a ring. Let $GL(n, R)$ denote the general linear group consisting of all $n \times n$ invertible matrices over R, and let $GL(R)$ be the direct limit of the sequence $GL(1, R) \subset GL(2, R) \subset \cdots$. A matrix in $GL(R)$ is called *elementary* if it coincides with the identity except for a single off diagonal entry. The subgroup $E(R) \subset GL(R)$ generated by the elementary matrices is precisely the commutator subgroup of $GL(R)$ and the abelian quotient group $GL(R)/E(R)$ is, by definition, $K_1(R)$.

Definition 1.1. The *Dennis trace* homomorphism DT: $K_1(R) \to HH_1(R)$ is given as follows. Let $\alpha \in K_1(R)$ be represented by an invertible $n \times n$ matrix A. Then $\mathrm{DT}(\alpha) = T_1(A \otimes A^{-1})$.

Returning to the situation of a group G with a endomorphism ϕ, let W be a given subset of G_ϕ. Define

$$HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi; W) \equiv \bigoplus_{C \in G_\phi - W} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$$

regarded as a subgroup of $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$. It may happen in topological applications that trace$(A \otimes B) \in C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ is not a cycle, but that for an appropriate geometrically defined $W \subset G_\phi$ its C-component $[\text{trace}(A \otimes B)]_C \in C_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ is a cycle for all $C \notin W$. Then we write $T_1(A \otimes B; W)$ for the element of $HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi; W) \subset HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi)$ whose C-component is represented by $[\text{trace}(A \otimes B)]_C$ for each $C \in G_\phi - W$.

(B) Review of Classical Fixed Point Theory

The one-parameter fixed point theory summarized in (C) will be easier to follow if the reader recalls the fundamentals of classical fixed point theory. References for all theorems in this subsection can be found in [9].

Let X be a finite connected CW complex and let $f: X \to X$ be a cellular map. Pick a basepoint $v \in X$, and a basepath τ in X from v to $f(v)$. Write $G = \pi_1(X, v)$ and let ϕ: $G \to G$ be the endomorphism induced by f; here it is understood that the path τ is to be used to get back from $f(v)$ to v. Pick an orientation for each cell of X so that we have a preferred basis for the cellular chains $C_*(X)$. Choose a lift, $\tilde{e}$, in the universal cover, $\tilde{X}$, of X for each cell e of

X. Orient $\tilde{e}$ compatibly with e. Let $\tilde{\tau}$ be the lift of τ which starts at the basepoint, $\tilde{v} \in \tilde{X}$, and let $\tilde{f}$ be the unique lift of f mapping $\tilde{v} \equiv \tilde{\tau}(0)$ to $\tilde{\tau}(1)$.

We regard $C_*(\tilde{X})$ as a right $\mathbb{Z}G$ chain complex; this means that if ω is a loop at v which lifts to a path $\tilde{\omega}$ starting at $\tilde{v}$, and if $h_{[\omega]}$ is the covering transformation sending $\tilde{v}$ to $\tilde{\omega}(1)$, then $h_{[\omega]}(\tilde{e}) = \tilde{e}[\omega]^{-1}$.

The chain map $\tilde{f}_*$: $C_*(\tilde{X}) \to C_*(\tilde{X})$ satisfies $\tilde{f}_k(\tilde{e}g) = \tilde{f}_k(\tilde{e})\phi(g)$, so it can be viewed as a chain map of right $\mathbb{Z}G$ complexes only when the right $\mathbb{Z}G$ action in the range is changed to $(\tilde{e}, g) \mapsto \tilde{e}\phi(g)$. Each $\tilde{f}_k$ is represented by a matrix $((\tilde{f}_k)_{ij})$ with entries in $(\mathbb{Z}G)^{\phi}$ defined by $\tilde{f}_k(\tilde{e}_j) = \sum_i \tilde{e}_i(\tilde{f}_k)_{ij}$. To represent all the $\tilde{f}_k$ together by a single matrix, we reuse the symbol $\tilde{f}_*$ to denote the matrix of $\oplus_k(-1)^k \tilde{f}_k$: $\oplus_k C_k(\tilde{X}) \to \oplus_k C_k(\tilde{X})$.

Then we have $T_0(\tilde{f}_*) \in HH_0(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$. As explained above in (A), we may regard $T_0(\tilde{f}_*)$ as lying in $\mathbb{Z}G_{\phi}$. Write $T_0(\tilde{f}_*) = \sum_{C \in G_{\phi}} \iota(f, C)C$ where each $\iota(f, C) \in \mathbb{Z}$. This element of $\mathbb{Z}G_{\phi}$ is classically known as the *Reidemeister trace*, $R(f)$, of the map f: $X \to X$. The integers $\iota(f, C)$ are called *fixed point indices of f*. The number of nonzero fixed point indices is the *Nielsen number*, $N(f)$, of f. The sum $\sum_{C \in G_{\phi}} \iota(f, C)$ is the *Lefschetz number*, $L(f)$.

Next, we recall the well-known theorems of classical fixed point theory.

THEOREM 1.2 (Lefschetz fixed point theorem). *If $L(f) \neq 0$, then f has a fixed point.*

THEOREM 1.3 (Nielsen–Wecken fixed point theorem). *The map f has at least $N(f)$ fixed points.*

Theorem 1.7 below is a stronger version of this theorem.

Our definitions have been of an algebraic nature, using a CW complex structure on X and requiring that f be cellular. But there is a parallel topological theory, which we now review. For this, let f: $X \to X$ be an arbitrary self map of a compact ANR. (Note that a finite CW complex is such.) Let the base point v, the base path τ, the fundamental group $G = \pi_1(X, v)$ etc. be as before.

Two fixed points x and y of a map f are *in the same fixed point class* if and only if for some path ν from x to y, the loop $\nu(f \circ \nu)^{-1}$ is homotopically trivial. This defines an equivalence relation on the set of fixed points, $\mathrm{Fix}(f)$. There is an injective function Ψ from the set of fixed point classes of f to the set, G_{ϕ}, of semiconjugacy classes: the class containing x is mapped to the semiconjugacy class containing $[\mu(f \circ \mu)^{-1}\tau^{-1}]$, where μ is any path from the basepoint v to x. There are only finitely many fixed point classes because nearby fixed points are clearly in the same class.

Each fixed point class B of f has a *fixed point index* $\iota(f, B) \in \mathbb{Z}$. The precise definition is given in [1, p. 56]; except in a simple case given below, we will not need this (complicated) definition. If $B_1, \ldots, B_k$ are the fixed point classes of f, the *intersection invariant* is defined to be $\Theta(f) = \sum_{j=1}^{k} \iota(f, B_j)\Psi(B_j) \in \mathbb{Z}G_{\phi} \equiv HH_0(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$. This $\Theta(f)$ enjoys good homotopy invariance properties:

THEOREM 1.4 (Invariance).

(a) *If $f \simeq g$ rel $\{v\}$ then $\Theta(f) = \Theta(g)$.*

(b) *If h: $(X, v) \to (X', v')$ is a homotopy equivalence, and if f: $X \to X$ and f': $X' \to X'$ satisfy $hf \simeq f'h$ rel$\{v\}$, then $h_*(\Theta(f)) = \Theta(f')$.*

We mention Theorem 1.4 because of our interest in its one-parameter analog, Theorem 1.12 below.

In the manifold case, $\Theta(f)$ has a geometric interpretation. Let X be a PL n-manifold. Every map f: $X \to X$ is homotopic to a PL map whose graph is transverse to the graph of the identity. Assume f has this property. Then $\mathrm{Fix}(f)$ is a finite set. Each point $x \in \mathrm{Fix}(f)$ has a *fixed point index* $\imath(f, x)$; it is defined to be the degree of the map $(\mathrm{id} - hfh^{-1})$: $N_\rho(h(x)) - \{h(x)\} \to \mathbb{R}^n - \{0\}$ where h is a homeomorphism from a neighborhood of $x \in X$ to a neighborhood $h(x) \in \mathbb{R}^n$ and N_ρ is a ball of radius $\rho > 0$ for suitably small ρ (see [1, p. 56] or [5, p. 146]). If the fixed point classes of f (which is transverse to the identity) are $B_1, \ldots B_k$ then the previously mentioned fixed point index reduces to $\imath(f, B_j) = \sum_{x \in B_j} \imath(f, x)$. Hence we get the geometric interpretation:

THEOREM 1.5. $\Theta(f) = \sum_{j=1}^{k}(\sum_{x \in B_j} \imath(f, x))\Psi(B_j)$.

We write $N(f)$ for the number of fixed point classes B_j for which $\imath(f, B_j) \neq 0$. This is the *Nielsen number* of f as defined in [1]. We will see shortly that it equals the previously defined Nielsen number when both make sense.

THEOREM 1.6. *If dim $X \geq 3$, every map $f: X \to X$ is homotopic to a map having exactly $N(f)$ fixed points.*

Our review is completed by stating the relationship between the algebraic and geometric invariants:

THEOREM 1.7. *Let $f: X \to X$ be a cellular map on a finite CW complex. Then $R(f) = \Theta(f)$. Hence the two definitions of $N(f)$ agree, and $R(f)$ is a homotopy invariant in both senses of Theorem* 1.4.

(*C*) *One-parameter Fixed Point Theory*

Proofs of all theorems in this subsection can be found in [9]. Let X be as in (B) and let $I = [a, b]$, where $a < b$, be a closed interval with the usual CW structure (consisting of two 0-cells and one 1-cell) and orientation of cells. Let $F: X \times I \to X$ be a cellular homotopy, where $X \times I$ has the product CW structure and its cells are given the product orientation.

Associated with F is a chain homotopy $D_k: C_k(X) \to C_{k+1}(X)$; if e is an oriented k-cell of X, $D_k(e)$ is the $(k+1)$-chain $(-1)^{k+1} F_*(e \times I) \in C_{k+1}(X)$, where $e \times I$ is given the product orientation.

Remark on Sign Conventions. Let $E_{i,\varepsilon}$ be the face of the cube $I^n \equiv [a, b]^n$ obtained by holding the i-th coordinate fixed at b (respectively, a) if $\varepsilon = 1$ (respectively, 0). We define the corresponding incidence number, $[I^n: E_{i,\varepsilon}]$, to be $(-1)^{i+\varepsilon}$. At the level of cellular n-chains we have $\partial_n I^n = \sum_i [I^n: E_{i,\varepsilon}] E_{i,\varepsilon}$. This convention leads to the above $(-1)^{k+1}$. Note that different sign conventions are used in [3].

With notation as in (B), we take $(v, a) \in X \times I$ as our basepoint and choose a basepath τ from v to $F(v, a)$. *We identify $\pi_1(X \times I, (v, a))$ with $\pi_1(X, v) \equiv G$ via the isomorphism induced by the projection $p: X \times I \to X$.* In particular, we write $\phi: G \to G$ for the homomorphism

$$\pi_1(X \times I, (v, a)) \xrightarrow{F_*} \pi_1(X, F(v, a)) \xrightarrow{C_{[\tau^{-1}]}} \pi_1(X, v).$$

Let $\tilde{\tau}$ be the lift of the basepath τ which starts at the basepoint, $\tilde{v} \in \tilde{X}$, and let $\tilde{F}$ be the unique lift of F mapping $(\tilde{v}, a)$ to $\tilde{\tau}(1)$. $\tilde{F}$ induces a chain homotopy $\tilde{D}_k$: $C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$. More precisely, $\tilde{D}_k(\tilde{e}) = (-1)^{k+1} \tilde{F}_k(\tilde{e} \times I)$. As in (*B*), this satisfies $\tilde{D}_k(\tilde{e}g) = \tilde{D}_k(\tilde{e})\phi(g)$. The boundary $\tilde{\partial}_k$: $C_k(\tilde{X}) \to C_{k-1}(\tilde{X})$, however, satisfies $\tilde{\partial}_k(\tilde{e}g) = \tilde{\partial}_k(\tilde{e})g$. Define endomorphisms of $\oplus_k C_k(\tilde{X})$ by $\tilde{D}_* = \oplus_k(-1)^{k+1}\tilde{D}_k$, $\tilde{\partial}_* = \oplus_k \tilde{\partial}_k$, $\tilde{F}_{a*} = \oplus_k(-1)^k \tilde{F}_{ak}$, and $\tilde{F}_{b*} = \oplus_k(-1)^k \tilde{F}_{bk}$. As in (B), we reuse the symbols $\tilde{D}_*$, $\tilde{\partial}_*$, $\tilde{F}_{a*}$, and $\tilde{F}_{b*}$ for the matrices of the corresponding endomorphisms. The chain homotopy relation yields the matrix equation:

$$\tilde{D}_*\phi(\tilde{\partial}_*) - \tilde{\partial}_*\tilde{D}_* = \tilde{F}_{a*} - \tilde{F}_{b*}.$$

The minus sign appearing on the left arises from our convention concerning the alternation of signs. Note that the entry of the matrix $\tilde{D}$ corresponding to an n-cell $\tilde{e}_1$ and an $(n+1)$-cell $\tilde{e}_2$ is the coefficient of $\tilde{e}_2$ in the $(n+1)$-chain $\tilde{F}_*(\tilde{e}_1 \times I)$.

Let $\omega = F(v \times I)$. Then $\tau\omega$ is a path from v to $F_1(v)$. Note that it is this path which must be used to determine the lift $\tilde{F}_b$ of F_b and so even if $F_a = F_b$ it is possible that $\tilde{F}_a \neq \tilde{F}_b$.

In the language of (A), consider trace$(\tilde{\partial}_* \otimes \tilde{D}_*) \in \mathbb{Z}G \otimes (\mathbb{Z}G)^{\phi}$. This is a Hochschild 1-chain whose boundary is

$$\operatorname{trace}(\tilde{D}_*\phi(\tilde{\partial}_*) - \tilde{\partial}_*\tilde{D}_*) = \operatorname{trace}(\tilde{F}_{a*} - \tilde{F}_{b*}).$$

The latter might not be zero, so trace$(\tilde{\partial}_* \otimes \tilde{D}_*)$ might not be a cycle; but in the important special case in which F_a and F_b have no fixed points then it can be shown ([9, Theorem 2.6]) that it is indeed a cycle. In this situation the invariant of interest to us is $T_1(\tilde{\partial}_* \otimes \tilde{D}_*) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$.

For the general case, recall that we are studying how Fix(F) is altered by homotopies of F rel $X \times \{a, b\}$. Since this is a relative problem it makes sense to remove the influence of F_a and F_b. Let $G_{\phi}(\partial F)$ be the subset $\{C_1, \ldots, C_k\}$ of G_{ϕ} consisting of semiconjugacy classes associated to fixed points of F_a or F_b.

In the notation of (A),

$$T_1(\tilde{\partial}_* \otimes \tilde{D}_*;\, G_{\phi}(\partial F)) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi};\, G_{\phi}(\partial F)) \subset HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})$$

is the desired invariant. As explained in (A) there is a natural isomorphism:

$$HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi};\, G_{\phi}(\partial F)) \equiv \bigoplus_{C \in G_{\phi} - G_{\phi}(\partial F)} HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C$$

$$\cong \bigoplus_{C \in G_{\phi} - G_{\phi}(\partial F)} H_1(Z(g_C)).$$

When convenient, we identify these abelian groups via the above isomorphism.

By analogy with the classical case discussed in (B), we define the following basic invariant of one-parameter fixed point theory.

Definition 1.8. The one-parameter trace of F is

$$R(F) \equiv T_1(\tilde{\partial}_* \otimes \tilde{D}_*;\, G_{\phi}(\partial F)) \in \bigoplus_{C \in G_{\phi} - G_{\phi}(\partial F)} HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \subset HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}).$$

The C-component of $R(F)$ is denoted by $\iota(F, C) \in HH_1(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \simeq H_1(Z(g_C))$. We call it the *fixed point index* of F corresponding to $C \in G_{\phi}$; it can be viewed as an element of $H_1(\tilde{X}/Z(g_C))$. The number of nonzero fixed point indices is the *one-parameter Nielsen number* $N(F)$ of F.

The augmentation ε: $\mathbb{Z}G \to \mathbb{Z}$ can be viewed as a morphism ε: $(\mathbb{Z}G)^{\phi} \to \mathbb{Z}$ of $\mathbb{Z}G$–$\mathbb{Z}G$ bimodules where $\mathbb{Z}$ is given the trivial bimodule structure, or as a morphism ε: $\overline{(\mathbb{Z}G)^{\phi}} \to \overline{\mathbb{Z}}$ of left $\mathbb{Z}G$ modules where $\bar{\mathbb{Z}}$ is the trivial module. There is a commutative diagram:

$$\begin{tikzcd} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \arrow[r, "\varepsilon"] \arrow[d, "\cong"'] & HH_*(\mathbb{Z}G, \mathbb{Z}) \arrow[d, "\cong"'] \\ H_*(G, \overline{(\mathbb{Z}G)^{\phi}}) \arrow[r, "\varepsilon"] & H_*(G, \bar{\mathbb{Z}}). \end{tikzcd}$$

In particular, the augmentation induces a natural homomorphism $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \to H_*(G)$. The *one-parameter Lefschetz class*, $L(F)$, is defined to be the image of the Hochschild homology class $R(F) \equiv T_1(\tilde{\partial}_* \otimes \tilde{D}_*;\ G_{\phi}(\partial F))$ in $H_1(G)$ under this homomorphism. Note that $L(F) = \sum_{C \in G_{\phi} - G_{\phi}(\partial F)} j_C(\iota(F, C))$ where j_C: $H_1(Z(g_C)) \to H_1(G)$ is induced by the inclusion $Z(g_C) \subset G$.

Remark. In the above formulation, $L(F)$ arose from the trivial representation of G in the sense that the augmentation is the extension of the trivial representation to $\mathbb{Z}G \to \mathbb{Z}$. More generally, let S be a ring (typically, S would be the algebra of $n \times n$ complex matrices) and let $\rho: G \to S^*$ be a representation of G in the group of units of S such that $\rho\phi = \rho$. One can define the *ρ-twisted one-parameter Lefschetz class*, $L(F, \rho) \in HH_1(\mathbb{Z}G, S) \cong H_1(G, \bar{S})$, where S has the $\mathbb{Z}G$–$\mathbb{Z}G$ bimodule structure determined by ρ, to be the image of $R(F)$ under the induced homomorphism ρ_*: $HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \to HH_*(\mathbb{Z}G, S)$.

Two fixed points (x, t) and (y, t') of F are *in the same fixed point class* if and only if for some path ν from (x, t) to (y, t'), the loop $(p \circ \nu)(F \circ \nu)^{-1}$ is homotopically trivial. This defines an equivalence relation on the set of fixed points, $\mathrm{Fix}(F)$. Just as in the classical case, there is an injective function Ψ from the set of fixed point classes of F to the set, G_{ϕ}, of semiconjugacy classes: the class containing (x, t) is mapped to the semiconjugacy class C containing $g_C \equiv [(p \circ \mu)(F \circ \mu)^{-1}\tau^{-1}]$, where μ is any path from the basepoint (v, a) to (x, t). It is easy to check that Ψ is well-defined, that F has only finitely many fixed point classes, and that fixed points in the same path component of $\mathrm{Fix}(F)$ are in the same fixed point class. Indeed, with the same notation, let ω be a loop in $\mathrm{Fix}F \subset X \times [a, b]$ based at (x, t), and let $h = [\mu\omega\mu^{-1}] \in G$. Then it is a straightforward exercise with homotopies to prove:

PROPOSITION 1.9. The element h lies in the semicentralizer $Z(g_C)$. □

The principal general theorems in [9] are:

THEOREM 1.10 (One-parameter Lefschetz fixed point theorem). *Suppose $L(F) \neq 0$. Then every map homotopic to F rel $X \times \{a, b\}$ has a fixed point not in the same fixed point class as any fixed point in $X \times \{a, b\}$. In particular, if F_a and F_b are fixed point free, every map homotopic to F rel $X \times \{a, b\}$ has a fixed point.*

THEOREM 1.11 (One-parameter Nielsen–Wecken fixed point theorem). *Every map homotopic to F rel $X \times \{a, b\}$ has at least $N(F)$ fixed point classes other than the fixed point classes which meet $X \times \{a, b\}$. In particular, if F_a and F_b are fixed point free, the fixed point set of every map homotopic to F rel $X \times \{a, b\}$ has at least $N(F)$ path components.*

THEOREM 1.12 (Invariance).

(a) *Let F, H: $X \times I \to X$ be cellular; if $F \simeq H$ rel $X \times \{a, b\}$ then $R(F) = R(H)$.*

(b) *If h: $X \to X'$ is a cellular simple homotopy equivalence, and if F: $X \times I \to X$ and F': $X' \times I \to X'$ are cellular homotopies such that $hF \simeq F'(h \times 1)$ rel $X \times \{a, b\}$, then $h_{\dagger}R(F) = R(F')$.*

When X is a compact polyhedron, it is possible to extend the definition of $R(F)$ and $L(F)$ to arbitrary continuous homotopies $F: X \times I \to X$ by means of simplicial approximation as follows. Let K be a triangulation of X and let J be the standard triangulation of I (with two vertices and one 1-simplex). Consider a simplicial approximation $E: Q \to K$ to F where Q is a subdivision of $K \times J$ ($K \times J$ denotes a product triangulation obtained without adding more vertices). Since $|E|: |K| \times I \to |K|$ is cellular (for the cell structure determined by K), we may form $R(|E|)$ and $L(|E|)$.

THEOREM 1.13. *There exists a sufficiently fine triangulation, K, of X so that $R(F)$ and $L(F)$ are well-defined by $R(|E|)$ and $L(|E|)$, respectively, i.e. if K' is any subdivision of K and $E': Q' \to K'$ is a simplicial approximation to F then $R(|E|) = R(|E'|)$ and $L(|E|) = L(|E'|)$. Furthermore, if $F': X \times I \to X$ is a continuous homotopy such that $F \simeq F'$ rel $X \times \{a, b\}$ then $R(F) = R(F')$ and $L(F) = L(F')$.*

We now turn to the manifold case. Let M be a PL (respectively smooth) n-manifold. Every map $M \times I \to M$ is homotopic to a PL (respectively smooth) map F whose graph is transverse to the graph of the projection, and is such that $\mathrm{Fix}(F) \cap M \times \{t\}$ is finite for each $t \in I$. It can be arranged that the graph of F then consists of circles and arcs which, for all but finitely many values of t, cross $M \times \{t\}$ transversely. If (x, t) lies in such a circle (or arc) crossing $M \times \{t\}$ transversely, orient the circle (or arc) in the direction of positive time if $\imath(F_t, x)$ is positive, and in the other direction if it is negative; see [5, § 8 and § 11] for details.

We are only interested in the circles, indeed in circles not in the same fixed point class as any arc. Let V be such a circle, let $(x, t) \in V$, let μ be a path in $M \times [a, b]$ from (v, a) to (x, t), and let ω be the loop based at (x, t) obtained by traversing V once in the direction of its orientation. With notation as above, Proposition 1.9 implies that the corresponding element, h, of G lies in $Z(g_C)$. In this way, we associate with V an element of $H_1(Z(g_C)) \cong (Z(g_C))_{\mathrm{ab}}$. If there are two circles V_1 and V_2 in the same fixed point class, one reaches the same semicentralizer $Z(g_C)$ from both circles provided the path used for $(x_1, t_1) \in V_1$ is μ, and the path used for $(x_2, t_2) \in V_2$ is $\mu\nu$, where $p(\nu)F(\nu)^{-1}$ is homotopically trivial. One treats any (finite) number of circles similarly. Thus, for each fixed point class, B, of F which does not meet $M \times \{a, b\}$, we have defined an element $\imath(F, B) \in H_1(Z(g_C))$, where $g_C \equiv [p(\mu)F(\mu)^{-1}\tau^{-1}]$ represents the semiconjugacy class C, and $\Psi(B) = C$.

Remark. In fact one can always reduce to the case in which only one circle occurs in each fixed point class. See [4].

Let $B_1, \ldots B_k$ be the fixed point classes of F which do not meet $M \times \{a, b\}$. Then, for $j = 1, \ldots k$ we have $\imath(F, B_j) \in H_1(Z(g_{C_j}))$ where $C_j = \Psi(B_j)$.

Definition 1.14. The *transverse intersection invariant of F* is:

$$\Theta(F) = \sum_{j=1}^{k} \imath(F, B_j) \in \bigoplus_{C \in G_\phi} H_1(Z(g_C)) \cong HH_1(\mathbb{Z}G, (\mathbb{Z}G)^\phi).$$

The main geometric theorem of [9] can be paraphrased as:

THEOREM 1.15. *Let M be an oriented smooth or PL n-manifold. Suppose the smooth or PL map $F: M \times I \to M$ has the above transversality properties. Then $\Theta(F) = -R(F)$. Also, $\sum_{j=1}^{k} j_C(\imath(F, B_j)) = -L(F)$, where j_C is induced by the inclusion $Z(g_C) \subset G$.*

Remark. Proofs in [9] are given for the PL case, but they go through with the obvious changes in the smooth case.

The extent to which $\Theta(F)$ (or $R(F)$) is the total obstruction to removing the relevant circles of fixed points by a homotopy of F rel $M \times \{a, b\}$ is discussed in [9].

(*D*) *Homological Computations*

Recall that when f is cellular, the Lefschetz number, $L(f)$, can be computed from the induced homomorphism f_*: $C_*(X) \to C_*(X)$ on cellular chains: $L(f) = \sum_i(-1)^i \operatorname{trace}(f_i)$. There is a similar method for computing the one-parameter Lefschetz class $L(F)$, where F: $X \times I \to X$ is cellular, $I = [a, b]$. Let $A_1 : G \to H_1(G) \equiv G_{\mathrm{ab}}$ be the abelianization homomorphism; its natural extension to a ring homomorphism $\mathbb{Z}G \to \mathbb{Z}G_{\mathrm{ab}}$ will also be denoted by A_1. Let $A_2 : \mathbb{Z}G_{\mathrm{ab}} \to G_{\mathrm{ab}}$ be the natural homomorphism and let $A : \mathbf{Z}G \to G_{\mathrm{ab}}$ be the composite $A = A_2 A_1$. Let X be as in (C). Let $\bar{X}$ be the universal *abelian* cover of X (i.e. the covering space of X corresponding to $\ker A_1$). Let $\bar{\partial}_*$ be the boundary operator of the cellular chain complex $C_*(\bar{X})$ and let D_* be the chain homotopy determined by F.

PROPOSITION 1.16. *Suppose F has no fixed points in $X \times \{a, b\}$. Then*

$$L(F) = A(\operatorname{trace}(\tilde{\partial}_* D_*)) = A_2(\operatorname{trace}(\bar{\partial}_* D_*)) \in G_{\mathrm{ab}}. \qquad \square$$

When making use of the above formula, one should take note of the alternation of signs convention built into the block matrix D_* as described in (C).

Remark. In the special case of a "cyclic" homotopy, i.e. a self-homotopy $F: f \simeq f$, $L(F)$ can be computed (over a field of coefficients) at the level of homology, see [10]. This formula involves the cohomology algebra of X and the homomorphism $H_*(X \times S^1) \to H_*(X)$ determined by the cyclic homotopy F; it can be viewed as a one-parameter analog of the Hopf trace formula which asserts, in the classical case, that a chain level trace is equal to a homology trace.

§ 2. ONE-PARAMETER FIXED POINT THEORY AND FLOWS

In this section we apply the theory of § 1 to flows.

(*A*) *The Fuller Index and the Fuller Homology Class*

Let M^n be a closed connected oriented C^∞-manifold. Let a C^∞ vector field $\mathcal{X}$ be given on M, and let Φ: $M \times \mathbb{R} \to M$ be the flow obtained by integrating $\mathcal{X}$. The point $(x, t) \in M \times (0, \infty)$ is a *periodic point* if $\Phi(x, t) = x$. There are two kinds of periodic points:

(i) if $\Phi(\{x\} \times \mathbb{R}) = \{x\}$ then every point of $\{x\} \times (0, \infty)$ is called a *stationary point*;

(ii) if there is a least positive number q such that $\Phi(x, q) = x$, then the set $\{(\Phi(x, t), q) \mid 0 \le t \le q\}$ is called a *periodic orbit of multiplicity* 1. More generally, let $k = mq$, where m is a positive integer, and call the set $\{(\Phi(x, t), k) \mid 0 \le t \le k\}$ *a periodic orbit of period k and multiplicity m.*

Thus, a periodic point $(x, t) \in M \times (0, \infty)$ is either stationary or lies on a periodic orbit.

A *periodic set*, P, is a subset of $M \times (0, \infty)$ which is the union of periodic orbits. This set is *isolated* if it has a compact neighborhood in $M \times (0. \infty)$ (called an *isolating neighborhood*) containing no other periodic points.

In his fundamental paper [8], Fuller assigns an integral homology class $\Lambda(P) \in H_1(M \times (0, \infty)) \equiv H_1(M)$ to each compact periodic set P. When P is an isolated periodic orbit of period k write $P \equiv \{(\Phi(x, t), k) \mid 0 \leq t \leq k\}$. Then $\Lambda(P)$ is the homology class represented by the *index cycle*, i.e. the singular 1-cycle $\imath(f^m, x) \cdot \omega$. Here, $(x, k) \in P$, the *primitive loop* ω is the 1-cycle $[0, q] \to M$ given by $t \mapsto \Phi(x, t)$; an open $(n-1)$-disk, D, containing x and transverse to P is chosen by the "method of sections" together with a "first return map", f, taking a neighborhood of x in D back into D, where f has the form $f(y) = \Phi(y, t(y))$; and $\imath(f^m, x) \in \mathbb{Z}$ is the classical fixed point index of "the m-th return map", f^m, at its isolated fixed point $x \in D$. When P is the union of finitely many isolated periodic orbits P_i, then $\Lambda(\cup P_i)$ is defined to be $\sum_i \Lambda(P_i)$.

When P is an arbitrary compact isolated periodic set for which V is an isolating neighborhood, one applies [8, Lemma 3.1] to perturb the vector field $\mathscr{X}$ to a new vector field $\mathscr{X}'$ whose corresponding periodic set P' (for which V is again isolating) is the union of finitely many isolated periodic orbits; and then one defines $\Lambda(P)$ to be $\Lambda(P')$. Fuller proves that $\Lambda(P)$ is well-defined and is invariant under appropriate deformation of the vector field $\mathscr{X}$ ("continuation"); see [8, Theorem 1].

This $\Lambda(P)$ is the *Fuller homology class* of P. If P is a single orbit of multiplicity m, we may rewrite our representative cycle as $(\imath(f^m, x)/m) \cdot m\omega$. The rational number $\imath(f^m, x)/m$ is called the *Fuller index* of P. There is a consistent homotopically invariant way of extending this $\mathbb{Q}$-valued Fuller index to arbitrary compact isolated periodic sets so that it is additive on finite unions of periodic orbits; see [8, Theorem 2].

Remark. Let $\{\tau\}$ denote the integral homology class of the integral cycle τ. Then $\Lambda(P) = \imath(f^m, x)\{\omega\}$. If $\{\omega\}$ is a torsion element of $H_1(M)$, then it is, in general, not valid to write $\Lambda(P)$ in the form (Fuller index of P)$m\{\omega\}$.

A useful homological characterization of $\Lambda(P)$ has been given by Franzosa [6]. Let the compact neighborhood N of P be a submanifold of $M \times (0, \infty)$. Define $\phi\colon (N, \partial N) \to (M \times M, M \times M - \Delta(M))$ by $\phi(x, t) = (x, \Phi(x, t))$. Here, $\Delta(M)$ denotes the diagonal in $M \times M$. Let $\mu' \in H^n(M \times M, M \times M - \Delta(M))$ be the Thom class, and let δ denote the Lefschetz duality isomorphism. Define $\Lambda'(P)$ to be the image of μ' under the composition

$$H^n(M \times M, M \times M - \Delta(M)) \xrightarrow{\phi^*} H^n(N, \partial N) \xrightarrow{\delta} H_1(N)$$
$$\xrightarrow{i_*} H_1(M \times (0, \infty)) \xrightarrow{p_*} H_1(M).$$

Here, i is inclusion and p is projection.

PROPOSITION 2.1. ([6; Prop. 3.2]) $\Lambda'(P) = \Lambda(P)$. □

By applying this proposition, we can use the one-parameter Lefschetz class of § 1(C) to detect periodic orbits as follows. Assume Φ has no stationary points. Let $0 < \varepsilon < U < \infty$ be such that Φ has no periodic orbit of period ε or U. Let P be the union of all the periodic orbits whose periods lie in (ε, U).

PROPOSITION 2.2. *Let* $F\colon M \times [\varepsilon, U] \to M$ *be the restriction of* Φ *to* $M \times [\varepsilon, U]$. *Then* $\Lambda'(P) = -L(F)$.

We show that $\Lambda'(P)$ is a standard "intersection class", and we then use Theorem 1.15 to conclude that the latter is $-L(F)$.

Let $\mu \in H^n(M \times M)$ be the image of the Thom class, μ', under the homomorphism on cohomology induced by the inclusion of pairs $(M \times M, \emptyset) \subset (M \times M,\ M \times M - \Delta(M))$. Let $[M] \in H_n(M)$ be the fundamental class, and let $\Delta\colon M \to M \times M$ be the diagonal map. Then μ is the Poincaré dual of $\Delta_*[M]$.

Let W be a compact disk bundle neighborhood of $\Delta(M) \subset M \times M$. There are three relevant cap products:

$$H^n(M \times M) \otimes H_{2n}(M \times M) \xrightarrow{\ \cap\ } H_n(M \times M)$$

$$H^n(M \times M,\ M \times M - \Delta(M)) \otimes H_{2n}(M \times M,\ M \times M - \Delta(M)) \xrightarrow{\ \cap\ } H_n(M \times M)$$

$$H^n(W, \partial W) \otimes H_{2n}(W, \partial W) \xrightarrow{\ \cap\ } H_n(W).$$

As we have said, $\mu \cap [M \times M] = \Delta_*[M]$. Using the three cap products, and functoriality (see [18, 5.6.16]), one deduces:

LEMMA 2.3. *$\mu_W \cap [W] = \Delta_*^W[M] \in H_n(W)$, where $[W] \in H_{2n}(W, \partial W)$ is the fundamental class, $\mu_W \in H^n(W, \partial W) \cong H^n(W, W - \Delta(M))$ is the image of μ' and $\Delta_*^W[M] \in H_n(W)$ corresponds to $\Delta_*[M] \in H_n(M \times M)$.* □

Let $\bar{F}\colon (M \times [\varepsilon, U],\ M \times \{\varepsilon, U\}) \to (M \times M,\ M \times M - \Delta(M))$ be the map $(x, t) \mapsto (x, F(x, t))$. Pick a compact manifold neighborhood, N, of $\operatorname{Fix} F$ sufficiently small that there is a commutative diagram:

$$\begin{tikzcd}
(M \times [\varepsilon, U], M \times \{\varepsilon, U\}) \arrow[r, "\bar{F}"] & (M \times M,\ M \times M - \Delta(M)) \\
(N, \partial N) \arrow[u, "k"] \arrow[r, "F'"] & (W, W - \Delta(M)) \arrow[u, "j"]
\end{tikzcd}$$

In the next Lemma, "$\cdot$" denotes the homology intersection product, (i.e. the dual of cup products of duals):

LEMMA 2.4. *$\mu_W \cap F'_*[N] = \Delta_*^W[M] \cdot F'_*[N]$.*

Proof. Let δ denote Lefschetz duality.

$$\begin{aligned}
\Delta_*^W[M] \cdot F'_*[N] &= \delta(\delta^{-1}\Delta_*^W[M] \cup \delta^{-1}F'_*[N]) \\
&= \delta(\mu_W \cup \delta^{-1}F'_*[N]) \text{ by Lemma 2.3} \\
&= (\mu_W \cup \delta^{-1}F'_*[N]) \cap [W] = \mu_W \cap (\delta^{-1}F'_*[N] \cap [W]) \\
&= \mu_W \cap F'_*[N].
\end{aligned}$$ □

Proof of Proposition 2.2. Let $\pi\colon M \times M \to M$ be projection on the first factor.

$$\begin{aligned}
\Lambda'(P) &= p_* i_*(k^* \bar{F}^*(\mu') \cap [N]) = p_* i_*(F'^*(\mu_W) \cap [N]) \\
&= \pi_* j_* F'_*(F'^*(\mu_W) \cap [N]) = \pi_* j_*(\mu_W \cap F'_*[N]) \\
&= \pi_* j_*(\Delta_*^W[M] \cdot F'_*[N]) \text{ by Lemma 2.4,} \\
&= \pi_*(\Delta_*[M] \cdot \bar{F}_*[M \times [\varepsilon, U]]).
\end{aligned}$$

It remains to show that this last expression coincides with $-L(F)$. For economy of notation, we write $J \equiv [\varepsilon, U]$ and $Q \equiv M \times J \times M$. Define F'': $M \times J \to Q$ by $F''(x,t) = (x,t,F(x,t))$. Define π': $Q \to M \times M$ to be the projection $(x,t,y) \mapsto (x,y)$ and π'': $Q \to M$ to be the projection $(x,t,y) \mapsto x$. Then $\pi'' = \pi\pi'$ and $\bar{F} = \pi' F''$. Let i: $M \times J \to M$ be $i(x,t) = (x,t,x)$. Assume that the disk bundle neighborhood W has been chosen so that $\bar{F}(M \times \partial J) \subset M \times M - W$. Define codimension 0 submanifolds $\partial_1 Q$ and $\partial_2 Q$ of ∂Q by $\partial_1 Q = (\pi')^{-1}(W) \cap \partial Q$ and $\partial_2 Q = (\pi')^{-1}(\overline{M \times M - W}) \cap \partial Q$. Then, $(Q; \partial_1 Q, \partial_2 Q)$ is a manifold triad. Let δ_1: $H^n(Q, \partial_1 Q) \to H_{n+1}(Q, \partial_2 Q)$ and δ_2: $H^n(Q, \partial_2 Q) \to H_{n+1}(Q, \partial_1 Q)$ be the Lefschetz duality isomorphisms. Consider the graph of F, Graph$(F) \subset Q$, and the graph of the projection proj: $M \times J \to M$, Graph(proj) $\subset Q$. The oriented manifold with boundary Graph(F) represents the homology class $F''_*[M \times J] \in H_{n+1}(Q, \partial_2 Q)$; similarly, Graph(proj) represents $i_*[M \times J] \in H_{n+1}(Q, \partial_1 Q)$. By Theorem 1.15, $-L(F)$ is represented geometrically by the 1-cycle obtained by making Graph(F) transverse rel ∂Q to Graph(proj) and then projecting to M. The homology class of this cycle is given by:

$$\begin{aligned}
\pi''_*(i_*[M \times J] \cdot F''_*[M \times J]) &\equiv \pi''_*((\delta_2^{-1}(i_*[M \times J]) \cup \delta_1^{-1}(F''_*[M \times J])) \cap [Q]) \\
&= \pi''_*(\delta_2^{-1}(i_*[M \times J]) \cap F''_*[M \times J]) \\
&= \pi_* \pi'_*((\pi')^*(\mu') \cap F''_*[M \times J]) \\
&= \pi_*(\mu' \cap \bar{F}_*[M \times J]) \quad (\text{note } \bar{F}_* = \pi'_* F''_*) \\
&= \pi_*(\Delta_*(M) \cdot \bar{F}_*[M \times J])
\end{aligned}$$

where in the third line we have used the fact that $\delta_2^{-1}(i_*[M \times J]) = (\pi')^*(\mu')$ (recall $\mu' \in H^n(M \times M, M \times M - \Delta(M))$ is the Thom class). □

Combining Propositions 2.1 and 2.2 we get:

THEOREM 2.5. *With Φ, F and P as above, the Fuller homology class $\Lambda(P)$ and the one-parameter Lefschetz class $L(F)$ are related by $\Lambda(P) = -L(F)$.* □

Remark. A proof of Theorem 2.5 which avoids reference to [6] and to the methods used above in the proofs of 2.2–2.4 can be constructed using the i_1-index of [4]; however, the dependence on Theorem 1.15 remains: compare (B) below.

To make computations from this, one should invoke Proposition 1.16:

COROLLARY 2.6. *Suppose Φ has no stationary points. Let Y be a CW decomposition of M and let D_*: $C_*(Y) \to C_*(Y)$ be a chain homotopy determined by $F = \Phi|_{M \times [\varepsilon, U]}$. Then the Fuller homology class of the union of all the periodic orbits in $M \times [\varepsilon, U]$ is:*

$$\Lambda(P) = -L(F) = -A(\operatorname{trace}(\tilde{\partial}_* D_*)) = -A_2\,(\operatorname{trace}(\bar{\partial}_* D_*)) \in H_1(M) \equiv G_{ab}.$$

In particular, if $A(\operatorname{trace}(\tilde{\partial}_* D_*)) \neq 0$ then Φ has a periodic orbit whose period lies in (ε, U). □

Of course the one-parameter Lefschetz class detects periodic orbits even when Φ has stationary points, but then F may have fixed points in $M \times \{\varepsilon, U\}$ so the clean formula in Proposition 1.16 may not be available.

(B) Fixed Point Classes of Orbits

The partition of the fixed points of Φ into fixed point classes was essentially explained in § 1(C), the only difference here being that the domain, $M \times \mathbb{R}$, is not compact, so that there might be infinitely many fixed point classes. However, with hypotheses and notation as in (A), the restriction $F = \Phi|_{M\times[\varepsilon,U]}$ has only finitely many fixed point classes. We wish to describe $R(F)$.

Pick a basepoint $x_0 \in M$. Since $\Phi_0 = \mathrm{id}$, we may assume that F induces the identity on G (see § 1(C)), so that the semiconjugacy classes of § 1(C) are true conjugacy classes.

Let P be an isolated periodic orbit of period k and multiplicity m, where $\varepsilon < k < U$. Let $x \in M$ be such that $(\Phi(x,t),k) \in P$ for some (hence all) t in $[0,k]$. As in (A), the primitive loop is ω: $[0,q] \to M$. Pick a path μ in M from x_0 to x. Let $C \subset G$ be the conjugacy class containing $g_C \equiv [\mu\omega^{-m}\mu^{-1}]$. By Proposition 1.9, $h \equiv [\mu\omega\mu^{-1}]$ determines an element of $H_1(Z(g_C))$ which we denote by $\{\omega\}$. We then associate with the orbit P the element $\imath(f^m,x)\{\omega\} \in H_1(Z(g_C)) \cong HH_1(\mathbb{Z}G)_C \subset HH_1(\mathbb{Z}G)$, where, as in (A), f is a first return map at $x \in P$.

Suppose the periodic set in $M \times (\varepsilon, U)$ consists of finitely many periodic orbits P_i (of multiplicity m_i, period $m_i q_i$, and defining primitive loop ω_i: $[0,q_i] \to M$). Let $C_i = \Psi(B(P_i))$ where $B(P_i)$ is the fixed point class of P_i (see § 1(C) for the definition of Ψ). Then $\{\omega_i\} \in H_1(Z(g_{C_i}))$ and we may form what we call the *Nielsen-Fuller invariant*:

$$\Theta'(F) = \sum_i \imath(f_i^{m_i}, x_i)\{\omega_i\} \in \bigoplus_{C\in G_\phi} H_1(Z(g_C)) \cong HH_1(\mathbb{Z}G),$$

where f_i is a first return map at $x_i \in P_i$. [By careful choice of the paths μ_i, one ensures that one gets the same $g_C \in C$ for all P_i corresponding to C.]

The similarity between this (non-transverse) invariant $\Theta'(F)$ and the transverse invariant $\Theta(F)$ defined in § 1(C) is intentional, for we have:

THEOREM 2.7. $-R(F) = \Theta'(F)$.

Proof. We make use of Theorem 4.4 of [4]. In that paper an integer index, i_1, is associated with each circle of fixed points. Inspection of the definition there shows that in the present situation this index leads to the invariant $\Theta'(\cdot)$, and in the transverse case it leads to $\Theta(\cdot)$. It is shown in [4] (see especially the proof of Proposition 3.6 of [4]) that F is homotopic rel $M \times \{\varepsilon, U\}$ to a map F' transverse to p such that the i_1-index is preserved. The meaning of this is that $\Theta'(F) = \Theta(F')$. By Theorem 1.15, $\Theta(F') = -R(F')$, and by Theorem 1.13, $R(F') = R(F)$. Note that while in [4] there is a standing hypothesis dim $M \geq 4$, that assumption is not needed for the argument being used here. □

Remarks.

(1) As we remarked in (A), the above proof also gives an alternative proof of Theorem 2.5.
(2) A similar result holds when Φ has stationary points. This involves "discarding" the conjugacy classes associated to F_ε and F_U as described in § 1(C).
(3) If one works over the rational numbers then $\Theta'(F) \otimes \mathbb{Q}$ can be rewritten as:

$$\Theta'(F) \otimes \mathbb{Q} = \sum_i (\text{Fuller index of } P_i) m_i (\{\omega_i\} \otimes 1) \in HH_1(\mathbb{Z}G) \otimes \mathbb{Q}.$$

§ 3. THE EULER CLASS OF THE NORMAL BUNDLE OF A NON-SINGULAR FLOW

In this section, we apply our methods to find a trace formula for the Euler class of the normal bundle of a non-singular flow.

The classical Poincaré–Hopf index theorem asserts that if $\mathscr{X}$ is a smooth vector field on the compact oriented manifold M with only finitely many zeros then the global sum of the indices of $\mathscr{X}$ equals the Euler characteristic of M. The global sum of the indices of $\mathscr{X}$ can be viewed as a geometric definition of the Poincaré dual of the Euler class of the tangent bundle of M; on the other hand the Euler characteristic can be defined as an alternating sum of traces on homology (or cellular chains). We now proceed to formulate and prove a "one-parameter" analog of the Poincaré–Hopf index theorem.

Let $\mathscr{X}$ be a non-singular vector field on the closed connected oriented n-manifold M and let Φ be the flow on M determined by $\mathscr{X}$. Let $\varepsilon > 0$ be such that the restriction F: $M \times [-\varepsilon, \varepsilon] \to M$ of Φ contains no fixed points other than the points of $M \times \{0\}$. (Of course, all points $(x, 0)$ are fixed.)

The vector field defines an oriented trivial line subbundle, λ, of $\tau \equiv \tau(M)$, the tangent bundle of M. Let η be any $(n-1)$-dimensional subbundle of τ complementary to λ, e.g. if M is given a Riemannian metric, take η to be the normal bundle of the flow Φ. The Euler class of the vector bundle η, denoted by $e(\eta)$, lies in $H^{n-1}(M)$, see [14].

Recall from § 1(C) the one-parameter Lefschetz class $L(F) \in H_1(M)$.

THEOREM 3.1 (One-parameter Poincaré–Hopf theorem). – *$L(F)$ is the Poincaré dual of $e(\eta)$.*

Proof. For any vector bundle γ, let $E_0(\gamma)$ denote the complement of the zero section of γ. Write $J = [-\varepsilon, \varepsilon]$. Let $\bar{F}$: $M \times (J, \partial J) \to (M \times M,\ M \times M - \Delta(M))$ be the map $(x, t) \mapsto (x, F(x, t))$, and let W be a compact disk bundle neighborhood of $\Delta(M) \subset M \times M$. To save notation we identify $(E(\tau), E_0(\tau))$ with $(\mathring{W},\ \mathring{W} - \Delta(M))$, using an exponential map, see [14, p. 121]. Define F', in the diagram below, to send (x, t) to the tangent vector at x of "length and direction t" in $E(\lambda) \subset E(\eta \otimes \lambda)$. The following diagram commutes up to homotopy (because F has no fixed points except at $M \times \{0\}$):

$$\begin{tikzcd}
(E(\eta \otimes \lambda), E_0(\eta \otimes \lambda)) \arrow[d, equal] & M \times (J, \partial J) \arrow[l, "F'"'] \arrow[dd, equal] \\
(E(\tau), E_0(\tau)) \arrow[d, "i"'] & \\
(M \times M,\ M \times M - \Delta(M)) & M \times (J, \partial J). \arrow[l, "\bar{F}"']
\end{tikzcd}$$

Here, i and j are inclusions.

The map F': $M \times J \to E(\eta \otimes \lambda)$ is homotopic to the "suspension" of the zero section s_0: $M \to E(\eta)$. Writing r for "restriction" and δ for duality, we obtain a commutative diagram:

$$\begin{tikzcd}
H^{n-1}(E(\eta), E_0(\eta)) \arrow[r, "(rs_0)^*"] \arrow[d, "\text{suspension}"'] & H^{n-1}(M) \arrow[r, "\delta"] \arrow[d, "\text{suspension}"'] & H_1(M) \arrow[dd, equal] \\
H^n(E(\tau), E_0(\tau)) \arrow[d, "(i^*)^{-1}"'] & H^n(\Sigma M) \arrow[d, "\cong"'] & \\
H^n(M \times M,\ M \times M - \Delta(M)) \arrow[r, "\bar{F}^*"] & H^n(M \times (J, \partial J)) \arrow[r, "\delta"] & H_1(M)
\end{tikzcd}$$

In this diagram, the fundamental class $u_\eta \in H^{n-1}(E(\eta), E_0(\eta))$ goes to $e(\eta) \in H^{n-1}(M)$; see [14, p. 119]; furthermore, the Thom class $\mu' \in H^n(M \times M, M \times M - \Delta(M))$ goes to $I \equiv \pi_*(\Delta_*[M] \cdot \bar{F}_*[M \times J]) \in H_1(M)$, where we use the notation of § 2(A); and u_τ is the "suspension" of u_η. As explained in the proof of Proposition 2.2, $I = -L(F)$. □

Remarks.

(1) In the special situation of Theorem 3.1, the invariant $R(F)$ carries no more information than $L(F)$ as all of the fixed points of F are associated with the conjugacy class of the identity element of $\pi_1(M)$.

(2) In [9, § 5], $L(F)$ is computed in a variety of situations using cellular chains; for example, in connection with the flows arising from circle actions on 3-dimensional lens spaces. By Theorem 3.1, these calculations give a practical way of computing the Euler class of the normal bundle. Another (purely homological) method of computing $L(F)$ in case F is a cyclic homotopy (such as a circle action) is given in [10] (see the remark following Proposition 1.16).

§4. SUSPENSION FLOWS

(*A*) *Suspension Semiflows*

Let X be a connected CW complex with base point $x_0 \in X$. Given a continuous map $f: X \to X$, its *mapping torus*, denoted by $T(X,f)$, is the space obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for each $x \in X$. The image of $(x, s) \in X \times [0, 1]$ in $T(X,f)$ will be denoted by $[x, s]$. If f is cellular then $T(X,f)$ inherits a natural CW structure. Choose a basepath σ from x_0 to $f(x_0)$ and let $\theta: H \to H$ be the self homomorphism of $H \equiv \pi_1(X, x_0)$ determined by f and σ.

Let $Y = T(X,f)$. Choose $y_0 = [x_0, 0]$ as a basepoint for Y and let $G = \pi_1(Y, y_0)$. There is a canonical map of Y to the standard circle S^1 (realized as complex numbers of unit modulus) given by: $p_f: Y \to S^1$, $p_f([x, s]) = e^{2\pi i s}$. Suppose that f is a π_1-*equivalence*; i.e. the induced map $f_\#: \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is an isomorphism. Thus $\theta: H \to H$, defined above, is an automorphism. Then the group G is a semidirect product of H with $T \equiv \pi_1(S^1, 1)$; there is an exact sequence: $H \rightarrowtail G \twoheadrightarrow T$ where $H \rightarrowtail G$ is induced by the inclusion $X \hookrightarrow Y$, $x \mapsto [x, 0]$, and $G \twoheadrightarrow T$ is induced by p_f. Choose $t \in G$ projecting to a generator of T so that $\theta: H \to H$ is given by $h \mapsto tht^{-1}$.

Warning. In the case where $f(x_0) = x_0$ this convention means that t is represented by the loop $I \to T(X,f)$, $u \mapsto [x_0, 1 - u]$. We make this choice because we deal with right modules, and we prefer t rather than t^{-1} to appear in our matrices.

A *semiflow* on a space is a continuous action of the additive semigroup of non-negative real numbers. The mapping torus $Y = T(X,f)$ supports a natural semiflow, denoted here by $\Phi: Y \times [0, \infty) \to Y$, called the *suspension of f*. It is given by the formula: $\Phi([x, s], u) = [f^{[s+u]}(x), (s + u) \bmod 1]$ where $[s + u]$ is the integer part of $s + u$. If f is cellular and $[0, \infty)$ carries the usual CW complex structure then Φ is a cellular map.

Fix an integer $\ell \geq 1$. Let $\Gamma = \Phi|_{Y \times [0, \ell+1]}$. We wish to compute $R(\Gamma)$.

Since θ is an isomorphism, the universal cover, $\tilde{Y}$, of $Y = T(X,f)$ can be thought of as the mapping telescope of $\tilde{f}: \tilde{X} \to \tilde{X}$. Assume f is cellular. Then we have the following model, denoted by $C_*(\tilde{Y})$, for the cellular chain complex of $\tilde{Y}$. Let $(C_*(\tilde{X}), {}_X\tilde{\partial})$ be the cellular chain complex of $\tilde{X}$. Define $C_*(\tilde{Y})$ by

$$C_n(\tilde{Y}) = (C_{n-1}(\tilde{X}) \oplus C_n(\tilde{X})) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \tag{4.1}$$

where the right action of G on $C_n(\tilde{Y})$ is given as follows: if $ht^j \in G$ and $a \otimes t^i \in C_n(\tilde{Y})$ then $(a \otimes t^i)ht^j \equiv a\theta^i(h) \otimes t^{i+j}$. A choice of oriented lifts of the $(n-1)$-cells and the n-cells of X determine a finite $\mathbb{Z}G$ basis for the right $\mathbb{Z}G$-module $C_n(\tilde{Y})$. The matrix of the boundary operator ${}_Y\tilde{\partial}_{n+1}: C_{n+1}(\tilde{Y}) \to C_n(\tilde{Y})$ with respect to the given $\mathbb{Z}G$ bases is:

$$\begin{bmatrix} [{}_X\tilde{\partial}_n] & 0 \\ (-1)^{n+1}(I - [\tilde{f}_n]t) & [{}_X\tilde{\partial}_{n+1}] \end{bmatrix}$$

where $[{}_X\tilde{\partial}_n]$ is the matrix of ${}_X\tilde{\partial}_n$, $[\tilde{f}_n]$ is the matrix of $\tilde{f}_n$ and I is an identity matrix of the same size as $[\tilde{f}_n]$. For the calculations in this section, the reader should note the Remark on Sign Conventions in §1(C).

Let $\tilde{D}: C_*(\tilde{Y}) \to C_{*+1}(\tilde{Y})$ be the chain homotopy defined by Γ. The matrix for $\tilde{D}_n: C_n(\tilde{Y}) \to C_{n+1}(\tilde{Y})$ is

$$\begin{bmatrix} 0 & (-1)^{n+1}\sum_{i=0}^{\ell}([\tilde{f}_n]t)^i \\ 0 & 0 \end{bmatrix}.$$

THEOREM 4.2. *$R(\Gamma) \in HH_1(\mathbb{Z}G)$ is represented by the Hochschild 1-cycle*

$$\sum_{n \geq 0} (-1)^{n+1} \operatorname{trace}((I - [\tilde{f}_n]t) \otimes \sum_{i=0}^{\ell-1} ([\tilde{f}_n]t)^i).$$

Proof. Computing the Hochschild 1-chain $\operatorname{trace}(\tilde{\partial}_* \otimes \tilde{D}_*)$ using the above matrices, we obtain $\sum_{n \geq 0}(-1)^{n+1}\operatorname{trace}((I - [\tilde{f}_n]t) \otimes \sum_{i=0}^{\ell}([\tilde{f}_n]t)^i)$. Since Γ_0 and $\Gamma_{\ell+1}$ have fixed points, we must discard terms corresponding to $G_1(\partial\Gamma)$ (see §1(C)). In the present context, this means discarding tensors whose total exponent sum in the variable t is 0 or $\ell + 1$. Note that a term of the form $1 \otimes u$ is a Hochschild boundary because $1 \otimes u = d(1 \otimes 1 \otimes u)$. In particular, terms of the form $\operatorname{trace}(I \otimes ([\tilde{f}_n]t)^\ell)$ can be omitted and terms of the form $\operatorname{trace}(I \otimes I)$ can be retained without affecting the homology class of the resulting chain. □

If, in Theorem 4.2, we let $\ell \to \infty$, the limit of the matrix $\sum_{i=0}^{\ell-1}([\tilde{f}_n]t)^i$ can be viewed as the inverse of $I - [\tilde{f}_n]t$ in the appropriate algebraic context, and then the whole formula becomes the Dennis trace of the element of K-theory represented by the finite alternating product of the invertible matrices $I - [\tilde{f}_n]t$, $n \geq 0$. This observation motivates much of §5 and §6.

The next two subsections provide geometric motivation for the algebra of §5.

(B) Fixed Point Classes

The language of periodic orbits carries over from §2(A). In this case there are no stationary points, and periods of periodic orbits are always positive integers. Indeed, if we regard X as a subset of Y, each periodic orbit of period k can be described as $\{(\Phi(x, u), k) \mid 0 \leq u \leq k\}$ where $x \in X$. Writing m for the multiplicity, we have $k = mq$. The point $x \in X$ is a "period q point" of f, i.e. q is the least positive integer such that $f^q(x) = x$. Conversely, if x is a period q point of f, if $m \geq 1$ and if $k = mq$, then $\{(\Phi(x, u), k) \mid 0 \leq u \leq k\}$ is a periodic orbit of period k and multiplicity m.

The fixed points of any iterate, f^r, are partitioned into fixed point classes as described in §1(B), and these correspond to semiconjugacy classes in H_{θ^r}. On the other hand, the periodic orbits of Φ consist of fixed points of Φ in the sense of §1(C), and their fixed point classes correspond to conjugacy classes contained in G_1 (the set of conjugacy classes of G). Let $G_1(r)$ be the subset of G_1 consisting of all conjugacy classes represented by elements of G of the form ht^r where $h \in H$; the sets $\{G_1(r)\}$ partition G_1. The function $\beta: H_{\theta^r} \to G_1(r)$ induced by $h \mapsto ht^r$ is surjective. The cyclic group $\langle \theta \rangle$ of automorphisms of H generated by θ acts on the set H_{θ^r}: θ sends the semiconjugacy class of h to that of $\theta(h)$. In fact, β induces a bijection $H_{\theta^r}/\langle \theta \rangle \to G_1(r)$. The following proposition is well known and easy to prove:

PROPOSITION 4.3. *Let x be a fixed point of f^r, whose corresponding semiconjugacy class is $C \in H_{\theta^r}$. Then $\beta(C) \in G_1$ is the conjugacy class corresponding to the orbit $\{(\Phi(x, u), r) \mid 0 \leq u \leq r\}$.* □

This action of $\langle \theta \rangle$ on H_{θ^r} partitions the fixed point classes into $\langle \theta \rangle$-orbits. We say that $x, y \in \mathrm{Fix}(f^r)$ are *in the same fixed point class of f^r mod $\langle \theta \rangle$* if the fixed point classes of x and y lie in the same $\langle \theta \rangle$-orbit.

We saw in §1(A) that $HH_0(\mathbb{Z}H, \mathbb{Z}H^{\theta^r}) \cong \mathbb{Z}H_{\theta^r}$. As will be shown in §5(A), θ induces an action on $HH_0(\mathbb{Z}H, \mathbb{Z}H^{\theta^r})$. The quotient group of co-invariants of this action, denoted by $HH_0(\mathbb{Z}H, \mathbb{Z}H^{\theta^r})_\theta$, plays an important role in the algebra in §5. The group $HH_0(\mathbb{Z}H, \mathbb{Z}H^{\theta^r})_\theta$ is naturally isomorphic to the quotient $\mathbb{Z}[H_{\theta^r}/\langle \theta \rangle]$. Proposition 4.3 should be seen as the geometric explanation for the appearance of $HH_0(\mathbb{Z}H, \mathbb{Z}H^{\theta^r})_\theta$.

(C) The Intersection Invariants

With notation as in (A), take the underlying space of X to be a closed connected smooth oriented $(n-1)$-manifold, $N, f: N \to N$ to be an orientation preserving diffeomorphism, and $Y \equiv T(N, f)$ to be our n-manifold M. Then the associated semiflow is (part of) a smooth flow on M. In this subsection we assume that for each iterate f^r of f $(r \geq 1)$ the graph of f^r is transverse to the graph of the identity.

The intersection invariant (see Theorem 1.5) of f^r is:

$$\Theta(f^r) = \sum_{B(r)} \sum_{x \in B(r)} \imath(f^r, x) \Psi(B(r)) \in \mathbb{Z}H_{\theta^r}$$

where $\{B(r)\}$ is the set of fixed point classes of f^r. Note that each $B(r)$ is a finite set. In (B) we had a coarser partitioning of the fixed points into fixed point classes mod$\langle \theta \rangle$. A typical such class is denoted by $B(r)_\theta$. The image of $\Theta(f^r)$ is:

$$\bar{\Theta}(f^r) = \sum_{B(r)_\theta} \sum_{x \in B(r)_\theta} \imath(f^r, x) \bar{\Psi}(B(r)_\theta) \in \mathbb{Z}[H_{\theta^r}/\langle \theta \rangle].$$

Here, $\bar{\Psi}(B(r)_\theta) \in H_{\theta^r}/\langle \theta \rangle$ has the obvious meaning.

The *orbit* of $x \in \mathrm{Fix}(f^r)$ is the set $Q \equiv \{f^s(x) \mid 0 \leq s \leq r-1\}$. If q_Q is the cardinality of this orbit, then q_Q divides r: the integer $m_Q \equiv r/q_Q$ is the *multiplicity* of the orbit Q. Since f is a diffeomorphism, $\imath(f^r, y)$ takes the same value at all points $y \in Q$.

PROPOSITION 4.4.

$$\bar{\Theta}(f^r) = r \sum_{m \mid r} \sum_{Q \in \mathcal{O}(f^r, m)} \frac{\imath(f^r, x_Q)}{m} \langle h_Q \rangle$$

where $\mathcal{O}(f^r, m)$ is the set of orbits Q of f^r of multiplicity m, x_Q is a representative point of Q, and $h_Q \in H$ represents the element of $H_{\theta^r}/\langle \theta \rangle$ associated (via $\bar{\Psi}$) with Q.

Proof. All the points of Q are in the same fixed point class mod $\langle \theta \rangle$. So the contribution of Q to $\bar{\Theta}(f^r)$ is $\sum_{x \in Q} \iota(f^r, x)\langle h_Q \rangle = q_Q\, \iota(f^r, x_Q)\langle h_Q \rangle = r[\iota(f^r, x_Q)/m]\langle h_Q \rangle$. □

The suspension flow Φ of f has finitely many periodic orbits of any given (positive integer) period.

Fix a positive integer ℓ, and let $0 < \varepsilon < \ell < U < \ell + 1$. Let $F = \Phi|_{M \times [\varepsilon, U]}$. The set $\mathrm{Fix}(F)$ consists of finitely many periodic orbits P, each of period $\leq \ell$. Let $k_P = m_P q_P$ be the period of P, where m_P is its multiplicity. Clearly, the Nielsen–Fuller invariant of §2(B) is:

$$\Theta'(F) = \sum_{k=1}^{\ell} \sum_{m|k} \sum_{P \in \mathcal{O}(F,k,m)} \iota(f^k, x_P)\{\omega_P\} \in HH_1(\mathbb{Z}G). \tag{4.5}$$

Here, $x_P \in N \cap P$ is a representative point of P, $\mathcal{O}(F, k, m)$ is the set of periodic orbits P of period k and multiplicity m, and $\omega_P:[0, q_P] \to M \times [\varepsilon, U]$ is the primitive loop of P. The homology class $\{\omega_P\}$ lies in $HH_1(\mathbb{Z}G)_{\Psi(B(P))}$, where the conjugacy class $\Psi(B(P)) \in G_1$ is specified as follows:

PROPOSITION 4.6. *Let $\iota(f^k, x_P)\{\omega_P\}$ be a summand in the above formula for $\Theta'(F)$, for a given period k and multiplicity m. Let Q be the orbit of $x_P \in \mathrm{Fix}(f^k)$. Let $k[\iota(f^k, x_P)/m]\langle h_Q \rangle$ be the corresponding summand of $\bar{\Theta}(f^k)$ in Proposition 4.4. Then the conjugacy class $\Psi(B(P)) \in G_1$ is represented by $h_Q t^k \in G$. Moreover, this conjugacy class is represented geometrically by the loop ω_P^{-m} in M.*

Proof. This follows from 4.3, Proposition 1.9 and the discussion in §2(B). □

All this suggests a comparison of $\Theta'(F) \in HH_1(\mathbb{Z}G)$ with

$$\Theta''(F) \equiv \sum_{r=1}^{\ell} \bar{\Theta}(f^r) \in \bigoplus_{r=1}^{\ell} \mathbb{Z}H_{\theta^r}/\langle \theta \rangle.$$

By Theorem 2.7, $\Theta'(F)$ carries the information we are studying about the orbits of Φ in $M \times [\varepsilon, U]$. On the other hand $\Theta''(F)$ is entirely derivable from the classical Nielsen fixed point theory of the maps $f, f^2, \ldots, f^\ell$. The comparison of these two invariants will be done in §6(B) once the required algebra is developed; see Theorem 6.12.

§5 K-THEORY AND ZETA FUNCTIONS

(A) *"Non-commutative" Analogs of Zeta Function Identities*

In this subsection we construct a "non-commutative" version of the Weil zeta function in an appropriate algebraic context.

Let S be an associative ring with unit and $\theta: S \to S$ a ring homomorphism. We do not assume S is commutative; in typical applications S will be the integral group ring of a group H and θ will be induced by an automorphism of H.

Let (C, ∂) be a finitely generated chain complex of right S-modules such that each C_i is free with a given basis. Suppose $f_*: C \to C$ is a *θ-homomorphism*, i.e. a degree 0 homomorphism of the underlying graded abelian groups such that $f_i(mr) = f_i(m)\theta(r)$ for $m \in C_i$ and $r \in S$. Recall that the *Reidemeister trace* of f_*, which we denote by $R(f_*)$, is the element of $HH_0(S, S^\theta)$, the 0-th Hochschild homology of S with coefficients in the bimodule S^θ (see

§1(A)), represented by the alternating sum of traces: $\sum_j(-1)^j \operatorname{trace}[f_j]$ where $[f_j]$ is the matrix of f_j with respect to the given basis.

For each integer $n \geq 1$, the endomorphism $\theta: S \to S$ induces a chain endomorphism of $C_*(S, S^{\theta^n})$ given on k-chains by $\theta_\#(a_1 \otimes \cdots \otimes a_k \otimes m) = \theta(a_1) \otimes \ldots \otimes \theta(a_k) \otimes \theta(m)$ which in turn induces an endomorphism, also denoted by $\theta_\#$, of $HH_*(S, S^{\theta^n})$. Define

$$HH_0(S, S^{\theta^n})_\theta \equiv \operatorname{coker}(1 - \theta_\#: HH_0(S, S^{\theta^n}) \to HH_0(S, S^{\theta^n})).$$

For $n \geq 1$ the n-th iterate of f_*, denoted f_*^n, is a θ^n-homomorphism. Note that $R(f_*^n) \in HH_0(S, S^{\theta^n})$. Let $\bar{R}(f_*^n)$ denote the image of $R(f_*^n)$ in $HH_0(S, S^{\theta^n})_\theta$. Geometric motivation for the passage from $R(f_*^n)$ to $\bar{R}(f_*^n)$ has been given in §4(B).

Definition 5.1. The *Lefschetz–Nielsen series* of a θ-homomorphism $f_*: C \to C$ is given by

$$\mathrm{LN}(f_*) = (\bar{R}(f_*^n))_{n=1}^\infty \in \prod_{n=1}^\infty HH_0(S, S^{\theta^n})_\theta.$$

Define the *θ-twisted power series ring*, denoted by $S_\theta[[t]]$, as follows: elements of $S_\theta[[t]]$ are formal series $\sum_{i=0}^\infty u_i t^i$ where $u_i \in S$ and t is an indeterminate. Multiplication is defined by

$$\left(\sum_{i=0}^\infty u_i t^i\right)\left(\sum_{j=0}^\infty v_j t^j\right) = \sum_{k=0}^\infty \left(\sum_{i+j=k} u_i \theta^i(v_j)\right) t^k.$$

Let $f_*: C \to C$ be a θ-homomorphism and let n_i be the cardinality of the given basis of C_i. Let I_m denote an $m \times m$ identity matrix. The matrix $I_{n_i} - [f_i]t$ is invertible over $S_\theta[[t]]$; its inverse is given by $\sum_{j=0}^\infty ([f_i]t)^j$. We regard $I_{n_i} - [f_i]t$ as an element of the infinite general linear group over $S_\theta[[t]]$ and so we can define $\Delta(f_*) \in K_1(S_\theta[[t]])$ as follows:

Definition 5.2. For $f_*: C \to C$ as above, $\Delta(f_*) \in K_1(S_\theta[[t]])$ is the element represented by the finite alternating product $\prod_{i \geq 0} (I_{n_i} - [f_i]t)^{(-1)^{i+1}}$.

This subsection is mainly concerned with the relationship between $\Delta(f_*)$ and $\mathrm{LN}(f_*)$, see Theorems 5.6 and 5.10. We begin by showing that $\Delta(f_*)$ is independent of choice of basis under a mild hypothesis on S.

PROPOSITION 5.3. *Suppose S has the property that any two bases of a finitely generated free module have the same cardinality. Then $\Delta(f_*) \in K_1(S_\theta[[t]])$ is independent of the choice of bases of the C_i's.*

Remark. An integral group ring satisfies the hypothesis of Proposition 5.3, ([3, p.36]).

Proof. Let $[f_i]'$ be the matrix of f_i with respect to another basis of C_i. By the hypothesis on S, the change of basis matrix A is a square matrix of the same size as $[f_i]$ and $[f_i]' = A[f_i]\theta(A^{-1})$. Thus

$$I_{n_i} - [f_i]'t = I_{n_i} - A[f_i]\theta(A^{-1})t = I_{n_i} - A[f_i]tA^{-1} = A(I_{n_i} - [f_i]t)A^{-1}$$

and so $I_{n_i} - [f_i]t$ and $I_{n_i} - [f_i]'t$ represent the same element of $K_1(S_\theta[[t]])$. □

Let $M_S \subset S_\theta[[t]]$ be the two sided ideal generated by t and let U_S be the subgroup of $S_\theta[[t]]^*$, the multiplicative subgroup of units of $S_\theta[[t]]$, of elements of the form $1 + u$, $u \in M_S$. Let $ev: S_\theta[[t]] \to S$ be evaluation at $t = 0$; note that ev is a ring homomorphism.

PROPOSITION 5.4. *Let $\bar{U}_S$ be the image of the natural map $U_S \to K_1(S_\theta[[t]])$. Then $K_1(S_\theta[[t]]) \cong \bar{U}_S \oplus K_1(S)$. More precisely, the sequence*

$$0 \to \bar{U}_S \xrightarrow{i} K_1(S_\theta[[t]]) \xrightarrow{ev_*} K_1(S) \to 0$$

is split exact where $\bar{U}_S \xrightarrow{i} K_1(S_\theta[[t]])$ is inclusion, ev_* is induced by evaluation at zero and the splitting is given by the homomorphism $K_1(S) \xrightarrow{j} K_1(S_\theta[[t]])$ induced by the inclusion of rings $S \subsetneq S_\theta[[t]]$.

Proof. Clearly $ev_* i = 0$ and $ev_* j$ is the identity, so it suffices to show that $\ker(ev_*) \subset \bar{U}_S$. An element $u \in \ker(ev_*)$ can be represented by an invertible matrix B which has the following form: its diagonal entries lie in U_S and its off diagonal entries lie in M_S. Consider the matrix operations: multiply a row by an element of U_S, multiply a column by an element of U_S, add a multiple $\lambda \in M_S$ of one row to a different row, add a multiple $\lambda \in M_S$ of one column to a different column. Such operations preserve the form of B; furthermore, B can be transformed to a diagonal matrix over U_S by such operations. A diagonal matrix over U_S represents an element of $\bar{U}_S$. □

If A is an $n \times n$ matrix over S then the matrix $I_n - At$, which is invertible over $S_\theta[[t]]$, represents an element of $\bar{U}_S$ because $ev(I_n - At) = I_n$. In particular, the alternating product, $\Delta(f_*)$, of Definition 5.2 also lies in $\bar{U}_S$.

Remark. The element of $p_A \in \bar{U}_S$ represented by the matrix $I_n - At$ can be thought of as a "non-commutative" analog of the characteristic polynomial of linear algebra. Indeed, if $S = F$ is a field and $\theta = \mathrm{id}$ then p_A is identified via the determinant to $\det(I_n - At)$ which is related to the characteristic polynomial, $c_A(t)$, of A by the formal identity $t^n c_A(t^{-1}) = \det(I_n - At)$ in $F[t, t^{-1}]$.

Let $\mathrm{DT}: K_1(S_\theta[[t]]) \to HH_1(S_\theta[[t]])$ be the Dennis trace (see §1(A) for the definition of the Dennis trace). Then $\mathrm{DT}(\Delta(f_*))$ is represented by the Hochschild 1-chain:

$$\sum_i (-1)^{i+1} \operatorname{trace}((I_{n_i} - [f_i]t) \otimes (I_{n_i} - [f_i]t)^{-1}).$$

For $n \geq 0$ define $C_{m,n}(S)$ to be the abelian subgroup of $C_m(S_\theta[[t]], S_\theta[[t]])$, the group of Hochschild m-chains on $S_\theta[[t]]$ (see §1(A)), generated by tensors of the form $h_0 t^{i_0} \otimes \cdots \otimes h_m t^{i_m}$ where $h_k \in S$ and $n = i_0 + \cdots + i_m$. Note that $C_{*,n}(S)$ is a subcomplex of $C_*(S_\theta[[t]], S_\theta[[t]])$. Consider the natural chain map $C_*(S_\theta[[t]], S_\theta[[t]]) \to \prod_{n=0}^{\infty} C_{*,n}(S)$ given by

$$\left(\sum_{i_0=0}^{\infty} u_{i_0,0} t^{i_0}\right) \otimes \cdots \otimes \left(\sum_{i_m=0}^{\infty} u_{i_m,m} t^{i_m}\right) \mapsto \left(\sum_{i_0 + \cdots + i_m = n} u_{i_0,0} t^{i_0} \otimes \cdots \otimes u_{i_m,m} t^{i_m}\right)_{n=0}^{\infty},$$

where $u_{i_k,k} \in S$.

Definition 5.5. $\widehat{HH}_*(S_\theta[[t]]) = H_*(\prod_{n=0}^{\infty} C_{*,n}(S))$.

The above chain map induces a homomorphism $HH_*(S_\theta[[t]]) \to \widehat{HH}_*(S_\theta[[t]])$. Observe that $\widehat{HH}_*(S_\theta[[t]]) \cong \prod_{n=0}^{\infty} H_*(C_{*,n}(S))$.

Define a homomorphism $J_n: C_{1,n}(S) \to C_0(S, S^{\theta^n})$ as follows. Consider a generating chain $ht^a \otimes kt^b$ where $h, k \in S$ and $a + b = n$. Then $J_n(ht^a \otimes kt^b) = ah\theta^a(k)$ where we view $h\theta^a(k)$ as a 0-chain lying in $C_0(S, S^{\theta^n})$. Suppose $z = ut^a \otimes vt^b \otimes wt^c$ where $u, v, w \in S$ and $a + b + c = n$. Then $J_n(dz) = b(v\theta^b(w)\theta^{b+c}(u) - u\theta^a(v)\theta^{a+b}(w))$. Since $v\theta^b(w)\theta^{b+c}(u)$

$-u\theta^a(v)\theta^{a+b}(w)$ maps to zero in $HH_0(S, S^{\theta^n})_\theta$, it follows that J_n induces a homomorphism $j_n: H_1(C_{*,n}(S)) \to HH_0(S, S^{\theta^n})_\theta$.

Define $\hat{P}: \widehat{HH}_1(S_\theta[[t]]) \to \prod_{n=0}^{\infty} HH_0(S, S^{\theta^n})_\theta$ to be the composite:

$$\widehat{HH}_1(S_\theta[[t]]) \xrightarrow{\cong} \prod_{n=0}^{\infty} H_1(C_{*,n}(S)) \xrightarrow{\prod_n j_n} \prod_{n=0}^{\infty} HH_0(S, S^{\theta^n})_\theta.$$

Also, let $\hat{P}_+ : \widehat{HH}_1(S_\theta[[t]]) \to \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$ be the composite $\hat{P}_+ = \pi_+ \hat{P}$ where $\pi_+ : \prod_{n=0}^{\infty} HH_0(S, S^{\theta^n})_\theta \to \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$ is projection. Define $P_+ : HH_1(S_\theta[[t]]) \to \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$ to be the composite of $HH_1(S_\theta[[t]]) \to \widehat{HH}_1(S_\theta[[t]])$ and $\hat{P}_+$.

THEOREM 5.6 ("Rationality" of the Lefschetz–Nielsen series). *For $f_*: C \to C$ as above:*

$$\mathrm{LN}(f_*) = P_+ \mathrm{DT}(\Delta(f_*)).$$

Proof. If A is a square matrix over S then $(At)^m = (\prod_{\ell=0}^{m-1} \theta^\ell(A))t^m$ for any integer $m \geq 0$. Furthermore, if B is another square matrix over S of the same size then $J_n(\mathrm{trace}(At^a \otimes Bt^b)) = a\ \mathrm{trace}(A\theta^a(B))$.

Recall that we denote the matrix of $f_i: C_i \to C_i$ by $[f_i]$. The chain

$$\sum_i (-1)^{i+1} \mathrm{trace}((I_{n_i} - [f_i]t) \otimes (I_{n_i} - [f_i]t)^{-1})$$

representing $\mathrm{DT}(\Delta(f_*))$ is homologous to the chain c given by

$$c = \sum_i (-1)^i \mathrm{trace}([f_i]t \otimes (I_{n_i} - [f_i]t)^{-1}).$$

Using the series $(I_{n_i} - [f_i]t)^{-1} = \sum_{j=0}^{\infty} ([f_i]t)^j$, we observe that the projection of c to $C_{1,n}(S)$, denoted by c_n, is:

$$c_n = \sum_i (-1)^i \mathrm{trace}([f_i]t \otimes ([f_i]t)^{n-1}).$$

From the above observations we see that

$$J_n(c_n) = \sum_i (-1)^i \mathrm{trace}\left(\prod_{\ell=0}^{n-1} \theta^\ell([f_i]) \right).$$

Now the matrix of f_i^n is $\prod_{\ell=0}^{n-1} \theta^\ell([f_i])$ and thus $R(f_*^n)$ is represented by the chain d_n given by

$$d_n = \sum_i (-1)^i \mathrm{trace}\left(\prod_{l=0}^{n-1} \theta^l([f_i]) \right).$$

In particular, $J_n(c_n) = d_n$ and so $j_n(\mathrm{DT}(\Delta(f_*))) = \bar{R}(f_*^n)$ in $HH_0(S, S^{\theta^n})_\theta$. □

Now suppose that the ring S is also a rational vector space. Then the homomorphism $\hat{P}_+ : \widehat{HH}_1(S_\theta[[t]]) \to \prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$ has a right inverse which can be thought of as a formal "logarithm":

Definition 5.7. The *logarithm*, Lg: $\prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta \to \widehat{HH}_1(S_\theta[[t]])$, is the homomorphism given as follows. For $n \geq 1$, let $u_n \in HH_0(S, S^{\theta^n})_\theta$ be represented by the 0-chain $U_n \in C_0(S, S^{\theta^n})$. Then

$$\mathrm{Lg}((u_n)_{n=1}^{\infty}) = \text{homology class of} \left(\sum_{n=1}^{\infty} \frac{U_n}{n} t^n \right) \otimes 1.$$

LEMMA 5.8. Lg *is well-defined.*

Proof. Suppose that for $n \geq 1$ the 0-chain $U'_n \in C_0(S, S^{\theta^n})$ also represents u_n. Then $U_n - U'_n = dV_n + (1 - \theta_\#)(E_n)$ where $V_n \in C_1(S, S^{\theta^n})$ and $E_n \in C_0(S, S^{\theta^n})$. Write $V_n = \sum_i a_i \otimes b_i$. Then

$$(U_n t^n)\otimes 1 - (U'_n t^n)\otimes 1 = d\Bigg(\sum_i (a_i \otimes (b_i t^n)\otimes 1 - (b_i t^n)\otimes a_i \otimes 1)$$

$$+ t\otimes (E_n t^{n-1})\otimes 1 - (E_n t^{n-1})\otimes t \otimes 1\Bigg).$$

Consequently, $(\sum_{n=1}^{\infty} \frac{1}{n} U_n t^n)\otimes 1 - (\sum_{n=1}^{\infty} \frac{1}{n} U'_n t^n)\otimes 1$ is a boundary. □

Observe the image of Lg lies in the image of $HH_1(S_\theta[[t]]) \to \widehat{HH}_1(S_\theta[[t]])$. It is straightforward to show:

PROPOSITION 5.9. $\hat{P}_+$ Lg *is the identity map of* $\prod_{n=1}^{\infty} HH_0(S, S^{\theta^n})_\theta$. □

The Hochschild homology element $\mathrm{DT}(\Delta(f_*)) \in HH_1(S_\theta[[t]])$ can be thought of as a substitute for the "Lefschetz zeta function" in the present "non-commutative" context; a suggestive name for it would be the "Reidemeister zeta element". This point of view is reinforced by the next theorem, a sharpened version of Theorem 5.6.

THEOREM 5.10 ("Rationality", second version). *Suppose S is a ring which is also a rational vector space. Then for* $f_*: C \to C$ *as in Theorem* 5.6:

$$\mathrm{Lg}(\mathrm{LN}(f_*)) = \widehat{\mathrm{DT}}(\Delta(f_*))$$

where $\widehat{\mathrm{DT}}$ *is the composite* $K_1(S_\theta[[t]]) \xrightarrow{\mathrm{DT}} HH_1(S_\theta[[t]]) \to \widehat{HH}_1(S_\theta[[t]])$.

Proof. From the proof of Theorem 5.6, the chain $d_n = \sum_i (-1)^i \operatorname{trace}(\prod_{l=0}^{n-1} \theta^l([f_i]))$ represents $\bar{R}(f_*^n) \in HH_0(S, S^{\theta^n})_\theta$. Observe that $d_n t^n = \sum_i (-1)^i \operatorname{trace}(([f_i]t)^n)$. The identity:

$$-\operatorname{trace}(([f_i]t)^n)\otimes 1 + n \operatorname{trace}([f_i]t \otimes ([f_i]t)^{n-1})$$

$$= d\left(\sum_{j=1}^{n-1} \operatorname{trace}([f_i]t \otimes ([f_i]t)^{n-j} \otimes ([f_i]t)^{j-1})\right)$$

implies that $(\frac{1}{n} \operatorname{trace}(([f_i]t)^n))\otimes 1$ is homologous to $\operatorname{trace}([f_i]t \otimes ([f_i]t)^{n-1})$. It follows that $\mathrm{Lg}(\mathrm{LN}(f_*))$ is represented by the chain $\sum_{n=1}^{\infty} \sum_i (-1)^i \operatorname{trace}([f_i]t \otimes ([f_i]t)^{n-1})$. The proof of Theorem 5.6 shows that this chain also represents $\widehat{\mathrm{DT}}(\Delta(f_*))$. □

Remark. By Proposition 5.9, applying the homomorphism $\hat{P}_+$ to the conclusion of the above theorem recovers the formula $\mathrm{LN}(f_*) = P_+\mathrm{DT}(\Delta(f_*))$ of Theorem 5.6 in the case the ring S is also a rational vector space.

Theorems 5.6 and 5.10 deserve to be called "rationality theorems" as they show that the Lefschetz–Nielsen series can be computed from a finite alternating product of "characteristic polynomials" defining an element of $K_1(S_\theta[[t]])$. Furthermore, in (B) below, we recover from Theorem 5.10 the familiar assertion that the Weil zeta function (in the topological context of [13]) and some other zeta functions used in dynamics and fixed point theory are rational functions.

(B) Change of Rings

In this subsection we show how the well known rationality of certain algebraic zeta functions used in fixed point theory and dynamics can be derived from the results of the previous subsection by means of a "change of rings". The main results are Theorem 5.13 and the examples which follow it.

As in §5(A), let S be a (possibly non-commutative) ring with an endomorphism θ. Let R be a *commutative* ring and let $\rho: S \to M_m(R)$ be a ring homomorphism such that $\rho\theta = \rho$ where $M_m(R)$ is the R-algebra of $m \times m$ matrices over R. In typical applications S will be the integral group ring of a group H, θ will be induced by an automorphism of H and ρ will be induced by a representation $H \to GL_m(R)$ where $GL_m(R)$ is the general linear group of $m \times m$ invertible matrices over R. The ring homomorphism ρ induces a homomorphism $\rho_*: HH_*(S, S^{\theta^n}) \to HH_*(M_m(R)) \cong HH_*(R)$ where the Morita equivalence isomorphism $HH_*(M_m(R)) \cong HH_*(R)$ is given explicitly by the trace; furthermore, since $\rho\theta = \rho$, it follows that ρ_* factors through $HH_*(S, S^{\theta^n})_\theta$ yielding a homomorphism $HH_*(S, S^{\theta^n})_\theta \to HH_*(R)$ which will also be denoted by ρ_*.

Let C be a finitely generated chain complex of right S-modules such that each C_i is free with a given basis, and let $f_*: C \to C$ be a θ-homomorphism. Let $C' = C \otimes_\rho M_m(R)$. Then C' is a finitely generated complex of free right $M_m(R)$-modules; furthermore, the given basis for C_i determines a basis for C'_i. Since $f_* \otimes \mathrm{id}: C' \to C'$ is a homomorphism of graded right $M_m(R)$-modules, where $\mathrm{id}: M_m(R) \to M_m(R)$ is the identity, we may form its Reidemeister trace:

$$R(f_* \otimes \mathrm{id}) \in HH_0(M_m(R)) \cong HH_0(R) = R.$$

Explicitly, $R(f_* \otimes \mathrm{id}) = \sum_j (-1)^j \operatorname{trace} \rho([f_j]) \in R$ where $\rho([f_j])$ is viewed as an $mn_i \times mn_i$ matrix over R and n_i is the cardinality of the given basis of C_i. It is clear that $\rho_*(\bar{R}(f_*)) = R(f_* \otimes \mathrm{id})$.

Notation. We write $L(f_*, \rho) \equiv R(f_* \otimes \mathrm{id})$. Following the terminology of [7], $L(f_*, \rho) \in R$ can be regarded as a "generalized Lefschetz number".

Examples 5.11. Let X be a finite connected CW complex with universal cover $\tilde{X}$ and fundamental group H. Let $g: X \to X$ be a cellular map. Let $S = \mathbb{Z}H$, the integral group ring of H. As in §1(B), g determines an endomorphism, θ, of H (and thus an endomorphism of S, also denoted by θ).

(1) Let $C \equiv C_*(\tilde{X})$ be the cellular chain complex of $\tilde{X}$ (regarded as a right S-module complex) and let $\tilde{g}_*: C_*(\tilde{X}) \to C_*(\tilde{X})$ be the induced θ-homomorphism (see §1(B)). Let $\rho: \mathbb{Z}H \to \mathbb{Z}$ be the augmentation homomorphism. Then $L(\tilde{g}_*, \rho) \in \mathbb{Z}$ is just the usual Lefschetz number, $L(g)$, of g (here $R = \mathbb{Z}$ and $m = 1$).

(2) Let A be an abelian group and $\rho: H \to A$ a surjective homomorphism such that $\rho\theta = \rho$. Let $\rho: \mathbb{Z}H \to \mathbb{Z}A$ also denote the extension of $\rho: H \to A$ to a homomorphism of group rings. Suppose $Y \subset X$ is a subcomplex and that $g(Y) \subset Y$, i.e. g is a map of pairs. Let $\tilde{Y}$ be the inverse image of Y under the covering projection $\tilde{X} \to X$. Let $C \equiv C_*(\tilde{X}, \tilde{Y})$ be the relative cellular chain complex of the pair $(\tilde{X}, \tilde{Y})$ (regarded as a right S-module complex) and let $\tilde{g}_*: C_*(\tilde{X}, \tilde{Y}) \to C_*(\tilde{X}, \tilde{Y})$ be the induced θ-homomorphism. Then $L(\tilde{g}_*, \rho) \in \mathbb{Z}A$ is the "generalized Lefschetz number" of [7, §3] (here $R = \mathbb{Z}A$, $m = 1$).

We wish to relate the invariant $\Delta(f_*)$ to the present discussion.

There is an extension of $\rho: S \to M_m(R)$ to a homomorphism $\bar{\rho}: S_\theta[[t]] \to M_m(R[[t]])$, where $R[[t]]$ is the commutative ring of formal power series over R, via the formula

$\bar{\rho}(\sum_{j=0}^{\infty} u_j t^j) = \sum_{j=0}^{\infty} \rho(u_j) t^j$ (by a minor abuse of notation, we have made dual use of the symbol t).

The matrices $I_{mn_i} - \bar{\rho}([f_i])t$ are invertible over $R[[t]]$ and so we can make the following definition.

Definition 5.12. For f_* and ρ as above, $\Delta(f_*, \rho) \in K_1(R[[t]])$ is the element represented by the finite alternating product $\prod_{i \geq 0} (I_{mn_i} - \bar{\rho}([f_i])t)^{(-1)^{i+1}}$.

Let $\bar{\rho}_*: K_1(S_\theta[[t]]) \to K_1(R[[t]])$ be the composite

$$K_1(S_\theta[[t]]) \to K_1(M_m(R[[t]])) \xrightarrow{\cong} K_1(R[[t]])$$

where the first homomorphism is induced by $\bar{\rho}: S_\theta[[t]] \to M_m(R[[t]])$ and the second homomorphism is the Morita equivalence isomorphism. Clearly, $\bar{\rho}_*(\Delta(f_*)) = \Delta(f_*, \rho)$.

Recall that a derivation $D: S \to S$ of a ring S is a homomorphism of abelian groups such that $D(uv) = D(u)v + uD(v)$. For any commutative ring A there is a derivation D_t of $A[[t]]$ given by $D_t(\sum_{i=0}^{\infty} a_i t^i) = \sum_{i=1}^{\infty} i a_i t^{i-1}$. If A is a rational vector space then there is a formal integration homomorphism $I: A[[t]] \to A[[t]]$ defined by

$$I(\sum_{i=0}^{\infty} a_i t^i) = \sum_{i=0}^{\infty} \frac{a_i}{i+1} t^{i+1}$$

which is a right inverse for D_t (i.e. $D_t I$ is the identity) and also $ID_t(u) = u - u_0$, $u = \sum_{i=0}^{\infty} u_i t^i$. Let $U_A \subset A[[t]]$ be the multiplicative subgroup of $A[[t]]$ consisting of power series of the form $1 + \sum_{i=1}^{\infty} u_i t^i$ and also let $M_A \subset A[[t]]$ be the ideal of series of the form $\sum_{i=1}^{\infty} u_i t^i$. There is a formal logarithm which is a homomorphism $\log: U_A \to M_A$ defined by

$$\log\left(1 + \sum_{i=1}^{\infty} u_i t^i\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\sum_{i=1}^{\infty} u_i t^i\right)^n.$$

This logarithm has the property that $D_t \log(u) = D_t(u)u^{-1}$. The inverse to $\log: U_A \to M_A$ is the formal exponential $\exp: M_A \to U_A$ given by the formula:

$$\exp\left(\sum_{i=1}^{\infty} u_i t^i\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^{\infty} u_i t^i\right)^n.$$

Before stating the next theorem, we recall:

Standing hypotheses. Let S be a (possibly non-commutative) ring with an endomorphism θ. Let R be a commutative ring and let $\rho: S \to M_m(R)$ be a ring homomorphism such that $\rho\theta = \rho$. Let C be a finitely generated chain complex of right S-modules such that each C_i is free with a given basis, and let $f_*: C \to C$ be a θ-homomorphism.

THEOREM 5.13. *Suppose that the commutative ring R is also a rational vector space. Then*

$$\det(\Delta(f_*, \rho)) = \prod_{i \geq 0} \det(I_{mn_i} - \bar{\rho}([f_i]t)^{(-1)^{i+1}} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} L(f_*^n, \rho) t^n\right).$$

Proof. The derivation D_t determines a homomorphism $(D_t)_*: HH_1(R[[t]]) \to R[[t]]$ which is induced by the map on Hochschild 1-chains given by $u \otimes v \mapsto D_t(u)v$. Similarly, there is a homomorphism $(\widehat{D_t})_*: \widehat{HH}_1(R[[t]]) \to R[[t]]$ such that D_t is the composite $HH_1(R[[t]]) \to \widehat{HH}_1(R[[t]]) \xrightarrow{(\widehat{D_t})_*} R[[t]]$. Applying Theorem 5.10 to the graded homomorphism $f_* \otimes \mathrm{id}: C \otimes_\rho M_m(R) \to C \otimes_\rho M_m(R)$, we obtain $\mathrm{Lg}((L(f_*^n, \rho))_{n=1}^\infty) = \widehat{\mathrm{DT}}(\Delta(f_*, \rho))$.

Let A' be the subring of $R[[t]]$ consisting of elements of the form $x1 + u$ where $x \in \mathbb{Q}$ and $u \in M_R$. Since A'/M_R is a field, A' is a local ring and so the determinant $\det: K_1(A') \to (A')^*$ is an isomorphism. Since $\Delta(f_*, \rho)$ is in the image of the homomorphism $K_1(A') \to K_1(R[[t]])$, it follows that $\Delta(f_*, \rho)$ and $\det(\Delta(f_*, \rho))$ represent the same element in $K_1(R[[t]])$ and so $\mathrm{Lg}((L(f_*^n, \rho))_{n=1}^\infty) = \widehat{\mathrm{DT}}(\det(\Delta(f_*, \rho)))$. Thus

$$(\hat{D}_t)_* \mathrm{Lg}((L(f_*^n, \rho))_{n=1}^\infty) = (\hat{D}_t)_* \widehat{\mathrm{DT}}(\det(\Delta(f_*, \rho))) \tag{5.14}$$

The left side of this identity simplifies to $D_t(\sum_{n=1}^\infty \frac{1}{n} L(f_*^n, \rho) t^n)$ and the right side to

$$D_t(\det(\Delta(f_*, \rho)))\,(\det(\Delta(f_*, \rho)))^{-1} = D_t(\log \det(\Delta(f_*, \rho))).$$

Since $\sum_{n=1}^\infty \frac{1}{n} L(f_*^n, \rho) t^n$ and $\log \det(\Delta(f_*, \rho))$ both belong to M_R, and $ID_t|_{M_R}$ is the identity homomorphism of M_R, applying the formal integration operator I followed by the formal exponential to 5.14 yields the conclusion of the theorem. □

Example 5.15. Let K be a field of characteristic zero, $S = R = K$ and suppose both ρ and θ are the identity $K \to K$. Then the conclusion of Theorem 5.13 is the familiar zeta function formula of [13, §3]: $\prod_{i \geq 0} \det(\mathrm{id} - f_i t)^{(-1)^{i+1}} = \exp(\sum_{n=1}^\infty \frac{1}{n} L(f_*^n) t^n)$.

Example 5.16. Consider the situation of Example 5.11(2). Let $\rho': \mathbb{Z}H \to \mathbb{Q}A$ be the composite of $\rho: \mathbb{Z}H \to \mathbb{Z}A$ and the inclusion $\mathbb{Z}A \subsetneq \mathbb{Q}A$. Then applying Theorem 5.13 with $S = \mathbb{Z}H$, $R = \mathbb{Q}A$, $C = C_*(\tilde{X}, \tilde{Y})$, $f_* = \tilde{g}_*$, and ρ' (in place of ρ) yields precisely the conclusion of [7, Theorem 3]; in particular, the series $\exp(\sum_{n=1}^\infty \frac{1}{n} L((\tilde{g}_*)^n, \rho') t^n)$ (denoted by $\zeta_{\rho'}(t)$ in [7]) is rational.

§6. THE TOTAL LEFSCHETZ–NIELSEN INVARIANT

In this section we apply the algebra of §5 to define a new K_1-type invariant for a map.

(A) A K_1-Invariant for Self-maps

Let X be a connected finite CW complex and let $x_0 \in X$ be a basepoint. Let $f: X \to X$ be cellular map. Choose a basepath $\sigma: [0, 1] \to X$ from x_0 to $f(x_0)$. Let $H = \pi_1(X, x_0)$. Then the composite $\pi_1(X, x_0) \xrightarrow{f_\#} \pi_1(X, f(x_0)) \xrightarrow{\sigma_\#^{-1}} \pi_1(X, x_0)$ defines a self homomorphism of H which will be denoted by θ.

Let $(\tilde{X}, \tilde{x}_0)$ be the universal cover of (X, x_0). Orient the cells of X so that we have a preferred basis for the integral cellular chains $(C_*(X), \partial)$. Choose a lift, $\tilde{e}$, in $\tilde{X}$ for each cell e of X and orient $\tilde{e}$ compatibly with e. There is a unique lift, $\tilde{f}: \tilde{X} \to \tilde{X}$, of f such that $\tilde{f}(\tilde{x}_0) = \tilde{\sigma}(1)$ where $\tilde{\sigma}$ is the unique lift of σ such that $\tilde{\sigma}(0) = \tilde{x}_0$.

Regard the cellular chains $(C_*(\tilde{X}), \tilde{\partial})$ as a free right $\mathbb{Z}H$ chain complex with preferred basis $\{\tilde{e}\}$. Then $\tilde{f}$ induces a θ-chain map $\tilde{f}_*: C_*(\tilde{X}) \to C_*(\tilde{X})$.

Definition 6.1. The *total Lefschetz–Nielsen invariant of* $f: X \to X$ is the element of $K_1(\mathbb{Z}H_\theta[[t]])$, denoted $\Delta(f)$, given by $\Delta(\tilde{f}_*)$. See Definition 5.2.

By Proposition 5.3, $\Delta(f)$ is independent of the choice of the lifts and of the orientations of the cells of X.

Suppose that Z is obtained from X by an elementary expansion, i.e. $Z = X \cup e^{q-1} \cup e^q$ where the $(q-1)$-cell e^{q-1} and the q-cell e^q are not cells of X and there is a map $u: I^q \to X$ such that u is a characteristic map for e^q, $u|_{I^{n-1}\times\{0\}}$ is a characteristic map for e^{q-1}, and $u(\overline{\partial I^q - I^{q-1} \times \{0\}}) \subset X^{q-1}$. Any cellular $f: X \to X$ extends to a cellular map $Z \to X$, let $g: Z \to Z$ be the composition of any such extension with the inclusion $i: X \subsetneq Z$, Let $H' = \pi_1(Z, i(x_0))$ and $\theta': H' \to H'$ be the self homomorphism determined by g. Clearly, $i_\#: H \to H'$ is an isomorphism and $i_\# \theta = \theta' i_\#$; furthermore, $i_\#$ induces an isomorphism $i_*: K_1(\mathbb{Z}H_\theta[[t]]) \to K_1(\mathbb{Z}H'_{\theta'}[[t]])$.

LEMMA 6.2. *For f and g as above $i_*(\Delta(f)) = \Delta(g)$.*

Proof. Choose lifts of the cells of $X \subsetneq Z$ and of $\{e^{q-1}, e^q\} \subset Z$. Observe that for $i = q-1, q$ we have that $C_i(\tilde{Z}) = C_i(\tilde{X}) \oplus \tilde{e}_i \mathbb{Z}H'$, a direct sum of based $\mathbb{Z}H'$-complexes. Let $[\tilde{f}_q]$ be the matrix of $\tilde{f}_q$ and $[\tilde{g}_q]$ be the matrix of $\tilde{g}_q$. Then the matrix of $\tilde{g}_q$ has the form

$$[\tilde{g}_q] = \begin{bmatrix} i_\#[\tilde{f}_q] & u \\ 0 & 0 \end{bmatrix}$$

where $u \in (\mathbb{Z}H')^{n \times 1}$, $n = \operatorname{rank} C_q(\tilde{X})$. Then

$$I_{n+1} - [\tilde{g}_q]t = \begin{bmatrix} I_n - i_\#[\tilde{f}_q]t & ut \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_n - i_\#[\tilde{f}_q]t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_n & (I_n - i_\#[\tilde{f}_q]t)^{-1} ut \\ 0 & 1 \end{bmatrix}$$

Since a matrix of the form $\begin{bmatrix} I_n & * \\ 0 & 1 \end{bmatrix}$ is a product of elementary matrices, $I_{n+1} - [\tilde{g}_q]t$ and $I_n - i_\#[\tilde{f}_q]t$ represent the same element of $K_1(\mathbb{Z}H'_{\theta'}[[t]])$. The same consideration applies to $I_{m+1} - [\tilde{g}_{q-1}]t$ and $I_m - i_\#[\tilde{f}_{q-1}]t$, $m = \operatorname{rank} C_{q-1}(\tilde{X})$; also $[\tilde{g}_j] = i_\#[\tilde{f}_j]$ for $j \neq q-1, q$. □

Since the next four propositions are obtained by applying Lemma 6.2 together with the standard techniques of simple homotopy theory (see [3]), only abbreviated proofs will be given.

PROPOSITION 6.3 (Homotopy class invariance). *If $f: X \to X$ and $g: X \to X$ are homotopic cellular maps then $\Delta(f) = \Delta(g)$.* □

Proof. Let $E: X \times I \to X$ be a homotopy from f to g, let $E': X \times I \to X \times I$ be the corresponding "fat" homotopy, i.e. $E'(x, t) = (E(x, t), t)$ and let i_0, i_1 be the inclusion of X into the 0 and 1 ends of $X \times I$, respectively. If x_0 is the basepoint of X and σ is the basepath for f choose the basepath for g to be σ followed by $s \mapsto E_s(x_0)$. Both i_0 and i_1 factor as a sequence of elementary expansions and so by Lemma 6.2, $(i_0)_*(\Delta(f)) = \Delta(E') = (i_1)_*(\Delta(g))$. □

In particular, the above proposition allows us to define $\Delta(f)$ for any continuous $f: X \to X$ by taking a cellular approximation $f': X \to X$ to f and setting $\Delta(f) \equiv \Delta(f')$.

PROPOSITION 6.4 (Simple homotopy invariance). *Suppose $s: X \to Y$ is a simple homotopy equivalence and there is a homotopy commutative diagram of cellular maps:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{g} & Y \end{array}$$

Then $s_*(\Delta(f)) = \Delta(g)$.

Proof. Replace s by a diagram $X \to Z \leftarrow Y$ where the two arrows factor as sequences of elementary expansions. $\square$

PROPOSITION 6.5 (Combinatorial invariance). *Suppose X' is a subdivision of X and $f': X' \to X'$ is a cellular map homotopic to $f: X \to X$. Then $\Delta(f) = \Delta(f')$.*

Proof. Since the identity id: $X \to X'$ is a simple homotopy equivalence, the conclusion follows from Proposition 6.4. $\square$

Using Chapman's theorem (see [2]) that homeomorphisms are simple homotopy equivalences, one reaches the following stronger conclusion:

PROPOSITION 6.6 (Topological invariance). *Suppose $h: X \to Z$ is a homeomorphism. Then $h_*(\Delta(f)) = \Delta(hfh^{-1})$.* $\square$

Applying Theorem 5.6, we see that we can recover the classical Nielsen fixed point theory of the iterates of $f: X \to X$ up to the ambiguity of passing to coker$(1 - \theta_\#)$ (i.e. the quantities $\tilde{R}(f^n) \equiv \tilde{R}((\tilde{f}_*)^n) \in HH_0(ZH, (\mathbb{Z}H)^{\theta^n})_\theta)$ from $\Delta(f)$:

COROLLARY 6.7. $\mathrm{LN}(\tilde{f}_*) = (\tilde{R}(f^n))_{n=1}^{\infty} = P_+ \mathrm{DT}(\Delta(f))$. $\square$

We call $\mathrm{LN}(f) \equiv \mathrm{LN}(\tilde{f}_*)$ the *Lefschetz–Nielsen series* of the map $f: X \to X$.

Example 6.8. Suppose the map $f: X \to X$ is homotopic to the identity. Then $\Delta(f) \in K_1(\mathbb{Z}H[[t]])$ is represented by the invertible element $(1 - t)^{-\chi(X)} \in (\mathbb{Z}H[[t]])^*$ where $\chi(X)$ is the Euler characteristic of X. In particular, if $\chi(X) = 0$ then $\Delta(f) = 0$ for such f.

Finally, we relate the total Lefschetz–Nielsen invariant of a π_1-equivalence $f: X \to X$, where X is a finite connected CW complex, to the results of §4.

Let $\mathbb{Z}H_\theta[t]$ denote the *θ-twisted polynomial ring* over $\mathbb{Z}H$, i.e. the subring of elements of $\mathbb{Z}H_\theta[[t]]$ involving only finitely many powers of t. Let ℓ be a positive integer. Define $\mu_\ell: HH_*(\mathbb{Z}H_\theta[[t]]) \to HH_*(\mathbb{Z}H_\theta[t])$ to be the composite:

$$HH_*(\mathbb{Z}H_\theta[[t]]) \to \prod_{n=0}^{\infty} H_*(C_{*,n}(\mathbb{Z}H)) \xrightarrow{\text{projection}} \bigoplus_{n=1}^{\ell} H_*(C_{*,n}(\mathbb{Z}H)) \xrightarrow{\Sigma} HH_*(\mathbb{Z}H_\theta[t])$$

where the monomorphism Σ is the sum of the natural maps $H_*(C_{*,n}(\mathbb{Z}H)) \to HH_*(\mathbb{Z}H_\theta[t])$ for $n = 1, \ldots, \ell$; see §5(A) for the definition of $C^{*,n}(\mathbb{Z}H)$.

We wish to construct a homomorphism $P': HH_1(\mathbb{Z}G) \to \bigoplus_{n=-\infty}^{\infty} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$ similar to the homomorphism P_+ of §5(A). For any integer n define $C'_{m,n}(\mathbb{Z}H)$ to be the abelian subgroup of $C_m(\mathbb{Z}G, \mathbb{Z}G)$ (the Hochschild m-chains on $\mathbb{Z}G$) generated by tensors of the form $h_0 t^{i_0} \otimes \cdots \otimes h_m t^{i_m}$ where $h_k \in H$ and $n = i_0 + \cdots + i_m$. Note that $C'_{*,n}(\mathbb{Z}H)$ is a subcomplex of $C_*(\mathbb{Z}G, \mathbb{Z}G)$. (The difference between $C_{m,n}(\mathbb{Z}H)$ and $C'_{m,n}(\mathbb{Z}H)$ is that negative powers of t are permitted in the latter.) Define a homomorphism $J_n: C'_{1,n}(\mathbb{Z}H) \to C_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})$ as follows. Consider a basis chain $ht^a \otimes kt^b$ where $h, k \in H$ and $a + b = n$. Then $J_n(ht^a \otimes kt^b) = ah\theta^a(k)$ where we view $h\theta^a(k)$ as a 0-chain lying in $C_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})$. As in §5(A), J_n induces a homomorphism $j_n: H_1(C'_{*,n}(\mathbb{Z}H)) \to HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$. Define P' to be the composite:

$$HH_1(\mathbb{Z}G) \xrightarrow{\cong} \bigoplus_{n=-\infty}^{\infty} H_1(C'_{*,n}(\mathbb{Z}H)) \xrightarrow{\bigoplus_{n=-\infty}^{\infty} j_n} \bigoplus_{n=-\infty}^{\infty} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta.$$

Also, let $P_\ell : HH_1(\mathbb{Z}G) \to \bigoplus_{n=1}^{\ell} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$ be the composite $P_\ell = \pi'_\ell P'$ where $\pi'_\ell : \bigoplus_{n=-\infty}^{\infty} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta \to \bigoplus_{n=1}^{\ell} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$ is projection.

As in the proof of Theorem 5.6, the Dennis trace of the total Lefschetz–Nielsen invariant, $\Delta(f) \in K_1(\mathbb{Z}H_\theta[[t]])$, is represented by the Hochschild chain

$$\sum_i (-1)^{i+1} \operatorname{trace}((I_{n_i} - [\tilde{f}_i]t) \otimes (I_{n_i} - [\tilde{f}_i]t)^{-1})$$

$$= \sum_i (-1)^{i+1} \sum_{j=0}^{\infty} \operatorname{trace}((I_{n_i} - [\tilde{f}_i]t) \otimes ([\tilde{f}_i]t)^j).$$

It follows that $\mu_\ell(\mathrm{DT}(\Delta(f)) \in HH_1(\mathbb{Z}H_\theta[t])$ is represented by the chain

$$\sum_i (-1)^{i+1} \sum_{j=0}^{\ell-1} \operatorname{trace}((I_{n_i} - [\tilde{f}_i]t) \otimes ([\tilde{f}_i]t)^j).$$

Let $i_* : HH_1(\mathbb{Z}H_\theta[t]) \to HH_1(\mathbb{Z}G)$ be the homomorphism induced by the natural inclusion $i : \mathbb{Z}H_\theta[[t]] \hookrightarrow \mathbb{Z}G$. A comparison with Theorem 4.2 yields the following:

THEOREM 6.9. *Let X be a finite connected CW complex and suppose that the cellular map $f: X \to X$ is a π_1-equivalence. Let $\Phi: Y \times [0, \infty) \to Y$ be the suspension semiflow on the mapping torus $Y = T(X, f)$ and let $\Gamma = \Phi|_{Y \times [0, \ell+1]}$. Then*

$$i_* \mu_\ell(\mathrm{DT}(\Delta(f))) = R(\Gamma) \in HH_1(\mathbb{Z}G). \qquad \square$$

Remark. In view of Theorem 6.9 we can think of $i_* \mathrm{DT}(\Delta(f))$ as a rigorous substitute for "$R(\Phi|_{Y \times [0,\infty)})$" (which is not rigorously defined because the one-parameter trace was defined in §1(C) only for homotopies parametrized by a compact interval).

Observe that $P_\ell i_* \mu_\ell = \pi_\ell P_+$ where $P_+ : HH_1(\mathbb{Z}H_\theta[[t]]) \to \prod_{n=1}^{\infty} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$ was defined in §5(A) and π_ℓ is projection to the first ℓ factors of $\prod_{n=1}^{\infty} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})_\theta$. Applying P_ℓ to the conclusion of Theorem 6.9 and invoking Theorem 5.6 we have:

COROLLARY 6.10. $P_\ell R(\Gamma) = \sum_{r=1}^{\ell} (\bar{R}(f^r) \in \bigoplus_{r=1}^{\ell} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^r})_\theta$. $\square$

Since $(\mathbb{Z}G)^{\otimes n} \otimes \mathbb{Q}$ is naturally isomorphic to $(\mathbb{Q}G)^{\otimes n}$ and $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we have that there is a natural isomorphism $HH_*(\mathbb{Z}G) \otimes \mathbb{Q} \cong HH_*(\mathbb{Q}G)$. Applying Theorem 5.10 to the conclusion of Theorem 6.9 yields:

COROLLARY 6.11. *The quantities $\bar{R}(f^r)$, $r = 1, \ldots, \ell$, determine $R(\Gamma)$ over the rational numbers, more precisely:*

$$i_* \mu_\ell \operatorname{Lg}((\bar{R}(f^r) \otimes \mathbb{Q})_{r=1}^{\infty}) = R(\Gamma) \otimes \mathbb{Q} \in HH_1(\mathbb{Z}G) \otimes \mathbb{Q} \cong HH_1(\mathbb{Q}G)$$

where i_* and μ_ℓ have the obvious meaning. $\square$

(B) *Application to Intersection Invariants*

We now return to the situation in §4(C), with the same hypotheses and notation. We are to compare the Nielsen–Fuller invariant $\Theta'(F) \in HH_1(\mathbb{Z}G)$ with the invariant $\Theta''(F) \equiv \sum_{r=1}^{\ell} \bar{\Theta}(f^r) \in \otimes_{r=1}^{\ell} \mathbb{Z}H_{\theta^r}/\langle \theta \rangle$. Our result is:

THEOREM 6.12. *Let $f: N \to N$ be an orientation preserving diffeomorphism of a closed connected oriented $(n-1)$-manifold. Suppose the graphs of the iterates $f, f^2, \ldots, f^\ell$ are transverse to the graph of id_N. Let M be the mapping torus $T(N, f)$, let Φ be the suspension flow* *of f, and let $F = \Phi|_{M\times[\varepsilon,U]}$ where $0 < \varepsilon < \ell < U < \ell + 1$. Then*:

(a) $i_*\mu_\ell \mathrm{DT}(\Delta(f)) = -\Theta'(F)$;

(b) $P_\ell\Theta'(F) = -\sum_{r=1}^{\ell}\bar{\Theta}(f^r)$, *so that* $\Theta'(F)$ *determines* $\Theta''(F)$;

(c) $-i_*\mu_\ell \mathrm{Lg}((\bar{\Theta}(f^r)\otimes\mathbb{Q})_{r=1}^{\infty}) = \Theta'(F)\otimes\mathbb{Q} \in HH_1(\mathbb{Q}G)$, so that, over $\mathbb{Q}$, $\Theta''(F)$ *determines* $\Theta'(F)$.

Proof. This would follow from Theorem 6.9 and Corollaries 6.10 and 6.11 if f were cellular. We get the desired result through suitable approximations.

Pick a smooth triangulation, K, of N. The diffeomorphism f is homotopic to a PL map f' which is simplicial from a subdivision of K to K, implying that $f':K \to K$ is cellular. Let $E:N\times I \to N$ be a homotopy from f to f'. We build E as a concatenation $E' * E''$ where E' is an isotopy from f to a PL homeomorphism $f'':|K| \to |K|$ (see [15, p. 103]), and E'' is a PL homotopy from f'' to f'. Let $\bar{E}:N\times I \to N\times I$ be the "fat" homotopy $(x,t)\mapsto(E(x,t),t)$. Our construction of E ensures that the maping torus $Z \equiv T(N\times I, \bar{E})$ is a polyhedron. There are inclusions $M \equiv T(N,f) \hookrightarrow Z$, and $Y' \equiv T(K,f') \hookrightarrow Z$, where we now regard K as a CW complex; M and Y' are subpolyhedra of Z. Let $\Lambda: Z\times[0,\ell+1], \to Z$ be the restriction of the suspension flow of $\bar{E}$. Then Λ restricts to Γ on $M\times[0,\ell+1]$, and to, say, Γ' on $Y'\times[0,\ell+1]$.

Let $\bar{\Lambda}$ be a PL approximation to Λ, restricting to $\bar{\Gamma}$ and $\bar{\Gamma}'$ on the two ends; $\bar{\Lambda}$ is to be chosen to be homotopic to Λ with respect to the polyhedral triad $(Z; M, Y')$. If $\bar{\Lambda}$ is chosen sufficiently close to Λ, then repeated applications of Theorems 1.12(b) and 1.13 give $R(\Gamma') = R(\bar{\Gamma}') = R(\bar{\Lambda}) = R(\bar{\Gamma}) = R(\Gamma)$.

Identifying $HH_1(\mathbb{Z}G, \mathbb{Z}G; G_1(\partial\Gamma))$ with the obvious subgroup of $HH_1(\mathbb{Z}G)$ and applying Proposition 4.6, we have $R(\Gamma) = R(F)$. So Part (a) follows from Theorems 2.7 and 6.9. Part (b) follows from Corollary 6.10, Theorems 1.7 and 1.4. Part (c) follows from Corollary 6.11. □

It follows from Theorem 6.12(a) that the Dennis trace of $\Delta(f)$ contains all the information we are studying about the orbits of Φ in $M\times(0,\infty)$.

§7. THE NOVIKOV RING AND A "NON-COMMUTATIVE" ALEXANDER–FRIED QUOTIENT

In this section we define a torsion invariant for certain finite connected CW complexes whose fundamental groups are semidirect products with an infinite cyclic group. In the special case of a complex Y arising as the mapping torus of a self map $f:X \to X$ of a finite connected complex X which induces an isomorphism of fundamental groups, we relate this torsion invariant to the total Lefschetz–Nielsen invariant. Our torsion invariant can also be thought of as a "non-commutative" generalization of the "Alexander invariant" of [13] or of the "Alexander quotient" of [7].

First, we briefly recall how the torsion of an acyclic complex is defined (see [3]). Let S be a ring and $U \subset S^*$ a given subgroup of the units of S. Assume S has the property that any two bases of a finitely generated free module have the same cardinality. Declare two bases of a finitely generated free module to be equivalent if the change of basis matrix represents an element of $K_1(S)$ which lies in the image of the natural map $U \to K_1(S)$. Define $K_1^U(S) \equiv \mathrm{coker}(U \to K_1(S))$. Let (C, ∂) be a finitely generated chain complex of right modules over S such that each C_i is free with a given equivalence class of bases. Suppose that C is acyclic. Let $\delta: C \to C$ be a chain contraction, $C_{\mathrm{odd}} = \oplus_{i\ \mathrm{odd}} C_i$ and $C_{\mathrm{even}} = \oplus_{i\ \mathrm{even}} C_i$. The restriction of $\partial + \delta$ to C_{odd} is an isomorphism $C_{\mathrm{odd}} \to C_{\mathrm{even}}$ and so its matrix with respect to bases chosen from each of the given equivalence classes defines an element of $K_1(S)$. The image of this element in $K_1^U(S)$ is independent of the choice of representatives of the equivalence classes of bases; it is called the *torsion* of (C, ∂) with respect to U. For example, when $S = \mathbb{Z}G$ is an integral group ring and $U = \pm G$ then $K_1^U(S)$ is the Whitehead group $Wh_1(G)$ and torsion in this context is called *Whitehead torsion.*

Let Y be a finite connected CW complex and y_0 a basepoint. Let $G \equiv \pi_1(Y, y_0)$. Suppose we are given an epimorphism $\psi: G \to T$ where T is infinite cyclic. Then G becomes a semidirect product of H with T, i.e. there is an exact sequence: $H \rightarrowtail G \twoheadrightarrow T$ where $H \equiv \ker(G \twoheadrightarrow T)$. Choose $t \in G$ projecting to a generator of T and define $\theta: H \to H$ to be conjugation by t, i.e. $\theta(h) = tht^{-1}$.

Orient the cells of Y so that we have a preferred basis for the integral cellular chains $(C_*(Y), \partial)$. Choose a lift, $\tilde{e}$, in $\tilde{Y}$ for each cell e of Y and orient $\tilde{e}$ compatibly with e. This procedure gives each $C_i(\tilde{Y})$ a preferred equivalence classes of bases.

In order to formulate our results, we need some preliminary discussion of the ring $\widehat{\mathbb{Z}G}^+$ obtained from $\mathbb{Z}G$ by a process of "completion" with respect to ψ.

Definition 7.1. The *Novikov ring* of (G, ψ), denoted by $\widehat{\mathbb{Z}G}^+$, is given as follows. Elements of $\widehat{\mathbb{Z}G}^+$ are formal series $\sum_{i=r}^{\infty} u_i t^i$ where $u_i \in \mathbb{Z}H$ and multiplication is defined by

$$\left(\sum_{i=r}^{\infty} u_i t^i\right)\left(\sum_{j=s}^{\infty} v_j t^j\right) = \sum_{k=r+s}^{\infty} \left(\sum_{i+j=k} u_i \theta^i(v_j)\right) t^k.$$

Up to isomorphism, $\widehat{\mathbb{Z}G}^+$ is independent of the choice of $t \in G$ projecting to a generator of T. Note that $\mathbb{Z}G$ is naturally included in $\widehat{\mathbb{Z}G}^+$ as finite sums $\sum_{i=r}^{n} u_i t^i$; also $\mathbb{Z}H_\theta[[t]]$ is naturally included as a subring of $\widehat{\mathbb{Z}G}^+$. The ring $\widehat{\mathbb{Z}G}^+$ is the smallest ring containing $\mathbb{Z}H_\theta[[t]]$ and $\mathbb{Z}G$ as subrings. Let $\mathbb{Q}[[t, t^{-1}]]^+$ be the field of formal infinite sums $\sum_{i=r}^{\infty} q_i t^i$, $q_i \in \mathbb{Q}$. The ring $\widehat{\mathbb{Z}G}^+$ has the property that any two bases of a finitely generated free module have the same cardinality because there is a ring homomorphism $\widehat{\mathbb{Z}G}^+ \xrightarrow{\bar{\psi}} \mathbb{Q}[[t, t^{-1}]]^+$ given by $\sum_{i=r}^{\infty} u_i t^i \mapsto \sum_{i=r}^{\infty} \varepsilon(u_i) t^i$ where $\varepsilon: \mathbb{Z}H \to \mathbb{Z} \subset \mathbb{Q}$ is the augmentation.

Recall that for any left $\mathbb{Z}G$-module M the homology $H_*(Y, M)$ is, by definition, $H_*(C_*(\tilde{Y}) \otimes_{\mathbb{Z}G} M)$.

Definition 7.2. Let Y be a finite connected CW complex with basepoint y_0 and fundamental group $G \equiv \pi_1(Y, y_0)$. Suppose $\psi: G \to T$ is an epimorphism to an infinite cyclic group T. Assume $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$. The *torsion* of (Y, ψ), denoted by $\tau(Y, \psi)$, is the element of $K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ given by the torsion of the acyclic chain complex $C(\tilde{Y}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$.

For natural examples of spaces Y which satisfy the hypothesis of the above definition, see Proposition 7.5 below.

Next, we relate the torsion of Definition 7.2 to the torsion of a homotopy equivalence.

Suppose Z and Y are finite connected CW complexes and $h: Z \to Y$ is a homotopy equivalence. Let $z_0 \in Z$ be basepoint, $G' = \pi_1(Z, z_0)$ and $G = \pi_1(Y, h(z_0))$. Suppose $\psi: G \to T$ is a surjection. Let $\psi': G' \to T$ be the composite $\psi h_\#$. Denote the Whitehead torsion of the homotopy equivalence h by $\tau(h) \in Wh_1(G)$. Let $\widehat{\mathbb{Z}G}^+$ (respectively $\widehat{\mathbb{Z}G'}^+$) be the Novikov ring of (G, ψ) (respectively (G', ψ')). The inclusion of rings $j: \mathbb{Z}G \subsetneq \widehat{\mathbb{Z}G}^+$ induces a homomorphism $j_*: Wh_1(G) \to K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$. Suppose $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$ so that $\tau(Y, \psi)$ is defined. Since h is a homotopy equivalence, it follows that $H_*(Z, \widehat{\mathbb{Z}G'}^+) = 0$ and thus $\tau(Z, \psi')$ is also defined. The torsions $\tau(h)$, $\tau(Y, \psi)$, $\tau(Z, \psi')$ are related by:

THEOREM 7.3. $j_*(\tau(h)) = \tau(Y, \psi) - h_*(\tau(Z, \psi'))$. *In particular, if h is a simple homotopy equivalence then* $\tau(Y, \psi) = h_*(\tau(Z, \psi'))$.

Proof. By replacing the map $h: Z \to Y$ with the inclusion of Z into its mapping cylinder, we can assume without loss of generality that h is the inclusion of a subcomplex $Z \subset Y$. There is an exact sequence of chain complexes: $0 \to C_*(\tilde{Z}) \to C_*(\tilde{Y}) \to C_*(\tilde{Y}, \tilde{Z}) \to 0$. This sequence remains exact when tensored over $\mathbb{Z}G$ with $\widehat{\mathbb{Z}G}^+$ because for each $n \geq 0$, $0 \to C_n(\tilde{Z}) \to C_n(\tilde{Y}) \to C_n(\tilde{Y}, \tilde{Z}) \to 0$ is a split exact sequence of $\mathbb{Z}G$-modules. The torsion of $C_*(\tilde{Y}, \tilde{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$ is $j_*(\tau(h))$ whereas the torsions of $C_*(\tilde{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$ and $C_*(\tilde{Y}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$ are $h_*(\tau(Z, \psi'))$ and $\tau(Y, \psi)$, respectively. The conclusion follows from a standard application of the "additivity of torsion", the precise statement of which can be found in [3, (17.2)]. □

Next, we show how natural examples of spaces Y satisfying the hypotheses of Definition 7.2 arise from the mapping torus construction described in §4(A). Let $Y = T(X, f)$ where X is a finite connected CW complex and $f: X \to X$ a cellular map. Retaining the notation of §4(A), the complex $\hat{C}_* \equiv C_*(\tilde{Y}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$ with boundary operator denoted by $\hat{\partial}_*$ is acyclic; indeed, an explicit chain contraction $\hat{D}: \hat{C}_* \to \hat{C}_{*+1}$ can be constructed as follows. Let $\hat{D}_n: \hat{C}_n \to \hat{C}_{n+1}$ be the homomorphism whose matrix is given by:

$$\begin{bmatrix} 0 & (-1)^{n+1} \sum_{i=0}^{\infty} ([\tilde{f}_n] t)^i \\ 0 & 0 \end{bmatrix}. \tag{7.4}$$

Then $\hat{\partial}_{n+1} \hat{D}_n + \hat{D}_{n-1} \hat{\partial}_n$ is the identity map. This proves:

PROPOSITION 7.5. *Let $Y = T(X, f)$ where X is a finite connected CW complex and $f: X \to X$ is a π_1-equivalence. Let ψ be the homomorphism from the fundamental group of Y to the infinite cyclic group induced by $p_f: Y \to S^1$. Then $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$. In particular, $\tau(Y, \psi)$ is defined.* □

The total Lefschetz–Nielsen invariant of f and the torsion invariant of (Y, ψ) are related as follows. The inclusion of rings $i: \mathbb{Z}H_\theta[[t]] \hookrightarrow \widehat{\mathbb{Z}G}^+$ induces a homomorphism $i_*: K_1(\mathbb{Z}H_\theta[[t]]) \to K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$.

THEOREM 7.6. *Let $Y = T(X, f)$ where X is a finite connected CW complex and $f: X \to X$ is a π_1-equivalence. Let $\psi: \pi_1(Y, y_0) \to T$ be the homomorphism induced by $p_f: Y \to S^1$. Then $i_*(\Delta(f)) = -\tau(Y, \psi)$.*

Proof. Since Y is given as the mapping torus $T(X, f)$, we can use (4.1) as our model for the cellular chain complex of $\tilde{Y}$ and the explicit chain contraction $\hat{D}$ of (7.4) to compute the torsion. Let $\hat{C}_* \equiv C_*(\tilde{Y}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}^+$ with boundary operator denoted by $\hat{\partial}_*$. Then

$$\hat{C}_{\text{odd}} = \hat{C}_{\text{even}} = \left(\bigoplus_i C_i(\tilde{X}) \right) \otimes \mathbb{Z}[[t, t^{-1}]]^+$$

where $\mathbb{Z}[[t, t^{-1}]]^+$ is the ring of formal series of the form $\sum_{j=r}^{\infty} u_j t^j$, $u_j \in \mathbb{Z}$. The lifts of the cells of X give a $\widehat{\mathbb{Z}G}^+$ basis. With respect to this basis, the matrix of the homomorphism $\hat{\partial} + \hat{D}: \hat{C}_{\text{odd}} \to \hat{C}_{\text{even}}$ is:

$$\begin{bmatrix} -(I - [\tilde{f}_0] t) & {}_X\tilde{\partial}_1 & 0 & 0 \cdots \\ 0 & (I - [\tilde{f}_1] t)^{-1} & {}_X\tilde{\partial}_2 & 0 \cdots \\ 0 & 0 & -(I - [\tilde{f}_2] t) & {}_X\tilde{\partial}_3 \\ \vdots & \vdots & 0 & \ddots \end{bmatrix}.$$

Multiplying by the elementary matrix

$$\begin{bmatrix} I & (I-[\tilde{f}_0]t)^{-1}x\tilde{\partial}_1 & 0 & 0\cdots \\ 0 & I & -(I-[\tilde{f}_1]t)x\tilde{\partial}_2 & 0\cdots \\ 0 & 0 & I & (I-[\tilde{f}_2]t)^{-1}x\tilde{\partial}_3 \\ \vdots & \vdots & 0 & \ddots \end{bmatrix},$$

we obtain the block diagonal matrix:

$$\begin{bmatrix} -(I-[\tilde{f}_0]t) & 0 & 0 & 0\cdots \\ 0 & (I-[\tilde{f}_1]t)^{-1} & 0 & 0\cdots \\ 0 & 0 & -(I-[\tilde{f}_2]t) & 0\cdots \\ \vdots & \vdots & 0 & \ddots \end{bmatrix};$$

this matrix represents the same element in $K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ as $b = \Pi_i(I_{n_i} - [\tilde{f}_i]t)^{(-1)^i}$. By Definition 5.2, $\Delta(f)$ is represented by b^{-1}. □

Observe that the homomorphism $P_+\mathrm{DT}: K_1(\mathbb{Z}H_\theta[[t]]) \to \Pi_{n\geq 1} HH_0(\mathbb{Z}H, (\mathbb{Z}H)^{\theta^n})$ of §5(A) factors through the natural homomorphism $K_1(\mathbb{Z}H_\theta[[t]]) \to K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$. Combining Theorems 7.6 and 5.6 we conclude:

Corollary 7.7. $\mathrm{LN}(f) = P_+\mathrm{DT}(-\tau(T(X,f),\psi))$. □

Next, we will show how the torsion invariant of Definition 7.2 can be viewed as a "non-commutative" generalization of Fried's "Alexander quotient" as defined in [7].

Recall that the torsion of an acyclic complex is functorial in the following sense. Suppose that S and R are rings, $U \subset S^*$ and $V \subset R^*$ are subgroups of the corresponding groups of units and $\rho: S \to R$ is a ring homomorphism such that $\rho(U) \subset V$. Assume R has the property that any two bases of a finitely generated free module have the same cardinality (consequently S will have the same property). Suppose C is an acyclic complex of free finitely generated right S-modules with given equivalence classes of bases and that $C \otimes_\rho R$ is also acyclic. Then the equivalence classes of bases for C determine equivalence classes of bases for $C \otimes_\rho R$. The torsion of C, denoted by $\tau(C)$, and the torsion of $C \otimes_\rho R$, denoted by $\tau(C \otimes_\rho R)$, are related by $\rho_*(\tau(C)) = \tau(C \otimes_\rho R)$ where $\rho_*: K_1^U(S) \to K_1^V(R)$ is the induced map in K-theory.

Let Y be a finite connected CW complex with fundamental group G and suppose that $\psi: G \twoheadrightarrow T$ is a surjective homomorphism to an infinite cyclic group. Let P be a finitely generated abelian group and $\rho: G \to P$ a surjective homomorphim. Assume that ψ factors through ρ, i.e. there exists $\psi': P \to T$ such that $\psi = \psi'\rho$. Let $\bar{Y}$ be the covering space of Y, corresponding to $\ker(\rho) \subset G$. We briefly recall Fried's formulation (see [7, §5]) of the *Alexander quotient.* Let $C_*(\bar{Y})$ be the cellular chain complex of $\bar{Y}$. We regard $C_*(\bar{Y})$ as a complex of right modules over the ring $\mathbb{Z}P$. Let $\mathbb{Z}P_N$ be the localization of $\mathbb{Z}P$ at its multiplicative group of non-zero divisors.

Lemma 7.8. *The ring $\mathbb{Z}P_N$ is a product of fields.*

Proof. The ring $\mathbb{Z}P_N$ is also the localization of $\mathbb{Q}P$ at its multiplicative group of non-zero divisors. Let $F \subset P$ be the torsion subgroup and A a free abelian complement to F in P. Then $\mathbb{Q}P \cong \mathbb{Q}F \otimes \mathbb{Q}A$. Since $\mathbb{Q}F$ is a direct product of cyclotomic fields, $\mathbb{Q}P$ is a direct product of Laurent polynomial rings over cyclotomic fields. It follows that $\mathbb{Z}P_N$ is a direct product of function fields over cyclotomic fields. □

For example, if P is free abelian of rank n then $\mathbb{Z}P_N$ is the field of rational functions in n variables over $\mathbb{Q}$.

Fried's "Alexander group" is $\mathbb{Z}P_N^*/(\pm P)$. Lemma 7.8 implies that the determinant induces an isomorphism $\det: K_1^{\pm P}(\mathbb{Z}P_N) \to \mathbb{Z}P_N^*/(\pm P)$. When the homology $H_*(\bar{Y})$ is "vulnerable", i.e. $C' \equiv C_*(\bar{Y}) \otimes_{\mathbb{Z}G} \mathbb{Z}P_N$ is acyclic, Fried's Alexander quotient, denoted by $\mathrm{ALEX}_P(Y)$ is the element $-\tau(C') \in K_1^{\pm P}(\mathbb{Z}P_N)$. Actually, Fried prefers to regard $\mathrm{ALEX}_p(Y)$ as an element of $\mathbb{Z}P_N^*/(\pm P)$ by applying the determinant isomorphism but the above formulation is more convenient here. Assume that $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$ so that $\tau(Y, \psi) \in K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ is defined. Also assume $H_*(\bar{Y})$ is vulnerable so that $\mathrm{ALEX}_P(Y)$ is defined. Let $H = \ker(\psi)$ and $H' = \ker(\psi')$; note that $\rho(H) \subset H'$. The ring $\widehat{\mathbb{Z}P}^+$ consists of formal power series of the form $\sum_{i=r}^{\infty} v_i t^i$ where $v_i \in \mathbb{Z}H'$. Let $\bar{\rho}: \widehat{\mathbb{Z}G}^+ \to \widehat{\mathbb{Z}P}^+$ be the ring homomorphism induced by ρ; explicitly, $\bar{\rho}(\sum_{i=r}^{\infty} u_i t^i) = \sum_{i=r}^{\infty} \rho(u_i) t^i$. Let $\widehat{\mathbb{Z}P}_N^+$ denote the localization of $\widehat{\mathbb{Z}P}^+$ at its multiplicative group of non-zero divisors and $j: \widehat{\mathbb{Z}P}^+ \subsetneq \widehat{\mathbb{Z}P}_N^+$ the natural inclusion of rings. The composite $j\bar{\rho}$ will also be denoted by $\bar{\rho}$. If $C_*(\tilde{Y})$ is the cellular chain complex of the universal cover $\tilde{Y}$ of Y then there is an identification $C_*(\tilde{Y}) \otimes_\rho \mathbb{Z}P \cong C_*(\bar{Y})$ and so

$$C_*(\tilde{Y}) \otimes_\rho \widehat{\mathbb{Z}P}_N^+ \cong C_*(\tilde{Y}) \otimes_\rho \mathbb{Z}P \otimes_{\mathbb{Z}P} \widehat{\mathbb{Z}P}_N^+ \cong C_*(\bar{Y}) \otimes_{\mathbb{Z}P} \widehat{\mathbb{Z}P}_N^+$$
$$\cong C_*(\bar{Y}) \otimes_{\mathbb{Z}P} \mathbb{Z}P_N \otimes_{\mathbb{Z}P_N} \widehat{\mathbb{Z}P}_N^+.$$

By assumption, the complex $C' \equiv C_*(\bar{Y}) \otimes_{\mathbb{Z}P} \mathbb{Z}P_N$ is acyclic and hence contractible since C' is free over $\mathbb{Z}P_N$. Thus $C' \otimes_{\mathbb{Z}P_N} \widehat{\mathbb{Z}P}_N^+$ is also contractible. By the functoriality of torsion, it follows that $\bar{\rho}_*(\tau(Y, \psi)) = i_*(\tau(C'))$ where $i_*: K_1^{\pm P}(\mathbb{Z}P_N) \to K_1^{\pm P}(\widehat{\mathbb{Z}P}_N^+)$ is induced from the inclusion $i: \mathbb{Z}P_N \to \widehat{\mathbb{Z}P}_N^+$. We have proved:

THEOREM 7.9. *Let Y be a finite connected CW complex with fundamental group G and $\psi: G \to T$ a surjective homomorphism to an infinite cyclic group. Let $\rho: G \to P$ be a surjective homomorphism to a finitely generated abelian group P such that ψ factors through ρ. If $\tau(Y, \psi)$ and* $\mathrm{ALEX}_P(Y)$ are both defined then $\bar{\rho}_*(\tau(Y, \psi)) = -i_*(\mathrm{ALEX}_P(Y))$. □

Using Lemma 7.8, we see that i_* is injective and so:

COROLLARY 7.10. *$\tau(Y, \psi)$ determines* $\mathrm{ALEX}_P(Y)$ □

Remark. When $Y = T(X, f)$ where $f: X \to X$ is a π_1-equivalence of a finite connected CW complex, $\tau(Y, \psi)$ and $\mathrm{ALEX}_P(Y)$ are both defined (for the latter see Proposition 2 and the proof of Proposition 6 of [7]).

Theorem 7.9 and its corollary justify the assertion that $\tau(Y, \psi)$ is a "non-commutative" generalization of Fried's Alexander quotient.

The above theory can be applied to obtain a new invariant for classical fibered knots. Let M^3 be a homology 3-sphere (i.e. a closed 3-manifold such that $H_*(M^3) \cong H_*(S^3)$) and let $K \subset M^3$ be a fibered knot. Then the knot complement, $Y = M^3 - N(K)$, where $N(K)$ is the interior of a tubular neighborhood of K, has the structure of a fiber bundle over the circle and the fiber, X, is a compact orientable surface with one boundary component. Thus Y is the mapping torus of a homeomorphism $f: X \to X$ (called the *monodromy* of K). Proposition 7.5 implies that the torsion invariant $\tau(Y, \psi) \in K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$ is defined, where

G is the knot group of K and $\psi: G \to T$ is the abelianization homomorphism. By Theorem 7.9 and Corollary 7.10, $\tau(Y, \psi)$ determines the Alexander quotient and hence the ideal in $\mathbb{Q}[t, t^{-1}]$ generated by the Alexander polynomial of K (see [7, §5]). This motivates the following:

Definition 7.11. The *non-commutative Alexander invariant* of a fibered knot K in a homology 3-sphere M^3 is the torsion invariant $\tau(M^3 - N(K), \psi) \in K_1^{\pm G}(\widehat{\mathbb{Z}G}^+)$.

Combining Theorem 7.6, Corollary 6.7, and Corollary 7.7, we obtain:

COROLLARY 7.12. *The non-commutative Alexander invariant determines the classical Nielsen fixed point theory of the iterates of the monodromy of a fibered knot (up to the ambiguity of passing to coker$(1 - \theta_{\#})$.*

Problem. To what extent does the non-commutative Alexander invariant distinguish fibered knots?

Problem. Does Definition 7.11 make sense for non-fibered knots? In other words, if $Y = M^3 - K$ is a knot complement with group G and $\psi: G \to T$ is the abelianization homomorphism, when is it true that $H_*(Y, \widehat{\mathbb{Z}G}^+) = 0$?

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